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Renormalization of QED with planar binary trees

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Abstract. The Dyson relations between renormalized and bare photon and electron propagators $Z_3\bar{D}(q) = D(q)$ and $Z_2\bar{S}(q) = S(q)$ are expanded over planar binary trees. This yields explicit recursive relations for the terms of the expansions. When all the trees corresponding to a given power of the electron charge are summed, recursive relations are obtained for the finite coefficients of the renormalized photon and electron propagators. These relations significantly decrease the number of integrals to carry out, as compared to the standard Feynman diagram technique. In the case of massless quantum electrodynamics (QED), the relation between renormalized and bare coefficients of the perturbative expansion is given in terms of a Hopf algebra structure.

1 Introduction

Wightman made the comment [1] "Renormalization Theory has a history of egregious errors by distinguished savants. It has a justified reputation of perversity; a method that works up to 13th order in the perturbation series fails in the 14th order." Although renormalization theory is considered to be well understood, it is still a difficult subject plagued with considerable combinatorial complexity.

However, renormalization is not just a recipe to extract a finite part from an infinite integral. It was a guide to elaborate the theories of weak and strong interactions. It can be used to build consistent Lagrangians: the minimal coupling Lagrangian of scalar electrodynamics misses a quartic term which is reintroduced by renormalization [2]. Renormalization is also linked to the irreversibility of the macroscopic universe [3]. Other arguments in favor of renormalization are provided by Jackiw [4]. Although a rigorous proof of renormalization was given in the sixties, the mathematical structure underlying renormalization was discovered only in 1998 by Kreimer [5,6], and later developed by Kreimer and Connes [7–10].

When practitoners of QED calculate multiloop contributions to renormalized propagators, they are often struck by the many cancellations occurring in the calculation. These cancellations are partly due to the existence of the Ward identity $Z_1 = Z_2$, which provides a relation between self-energy and vertex counterterms. In this paper, we derive recursive equations for the renormalized electron and photon propagators that take full account of $Z_1 = Z_2$.

In Schwinger's approach, the electron and photon propagators are given as solutions of equations involving functional derivatives. In [11], it was shown that the Schwinger equations for QED can be solved elegantly using series indexed by planar binary trees: the electron and photon propagators were written as

$$S(q) = \sum_t e_0^{2|t|}\varphi^0(t;q),$$
$$D_{\lambda\mu}(q) = \sum_t e_0^{2|t|}\varphi^0_{\lambda\mu}(t;q),$$

where t runs over the set of planar binary trees, and $|t|$ is the number of internal vertices of t. Recurrence relations were obtained to calculate $\varphi(t)$ and $\varphi_{\lambda\mu}(t)$.

Schwinger also gave an equation for the renormalized propagators $\bar{S}(q)$ and $\bar{D}_{\lambda\mu}(q)$. If we enforce the relation $Z_1 = Z_2$, the Schwinger equations for the renormalized propagators contain only the renormalization factors Z_2 and Z_3, and the mass shift δm. The renormalized propagators are expanded over trees:

$$\bar{S}(q) = \sum_t e^{2|t|}\bar{\varphi}^0(t;q),$$
$$\bar{D}_{\lambda\mu}(q) = \sum_t e^{2|t|}\bar{\varphi}^0_{\lambda\mu}(t;q),$$

and the Schwinger equation is solved, giving us recurrence relations for $\bar{\varphi}(t)$ and $\bar{\varphi}_{\lambda\mu}(t)$. However, enforcing $Z_2 = Z_1$ is incompatible with renormalization at the tree level, in

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the sense that the $\bar{\varphi}(t)$ given by the recurrence relations are not finite. To obtain finite quantities, we make two partial sums. In the first case, we sum over all terms of a given order; in the second, we sum over all terms with a given number of electron and photon loops. This last case generalizes quenched QED (i.e. zero electron loop). Recurrence relations will be given for these terms. In the case of massless QED, the recursive relations will be transformed into the definition of the Hopf algebra of renormalization.

The plan of the paper is the following. We first introduce the problem of renormalization from different points of view; then we give the essential equations for the renormalization of QED. The tree-expansion method is presented in detail and used to derive the recursive equations for the renormalized electron and photon propagators. These relations will be summed over all trees of a given order, and over all trees with a given number of electron and photon loops. These sums will be shown to be finite. Finally the Hopf algebra of massless QED is described. A first appendix gives some proofs, a second one gives the relation between renormalized and bare propagators for massless QED. A last appendix expresses the smallest planar binary trees as a sum of Feynman diagrams.

In this paper, we consider only the renormalization of ultraviolet divergences, and we assume that infrared divergences are regularized, for instance by introducing a photon mass.

2 Renormalization

An enormous literature has been devoted to the renormalization theory. The reader is referred to [12–14] and references therein for the concepts and history of renormalization. Here we shall concentrate on its technical aspects. Renormalization theory can be considered from at least three different points of view: the Dyson method, the extension of distributions and the product of distributions.

2.1 The Dyson point of view

According to the first point of view, perturbative quantum field theory yields divergent integrals in Fourier space, and renormalization is a technique intended to extract a finite part from them. A picture of how this could be achieved was first given by Dyson in 1949 [15] and Salam [16,17] in 1951. Explicit formulas were proposed by Bogoliubov and Parasiuk [18] and finally proved by Hepp [19,20]. This method is general, in the sense that it can be used for any quantum field theory, whether renormalizable or not.

To understand this renormalization process, it is very useful to treat a one-dimensional toy model of overlapping divergences proposed by Kreimer [5,6]. Let

$$f(x,y,c) = \frac{x}{x+c}\frac{1}{x+y}\frac{y}{y+c}.$$

We want to give a meaning to the integral

$$I(c) = \int_1^\infty \mathrm{d}x \int_1^\infty \mathrm{d}y f(x,y,c).$$

Power counting is applied as follows. If we substitute λx for x and take the limit $\lambda \to \infty$, we see that $I(c)$ varies as $\lambda^0 = 1$, and we say that the integral is logarithmically divergent for x. Similarly, it is logarithmically divergent for y. If both x and y are multiplied by λ, the integral $I(c)$ varies as λ^1 in the limit $\lambda \to \infty$. Then, $I(c)$ is linearly divergent for the variables x, y. In the Dyson–Salam renormalization scheme, we first fix y in $f(x,y,c)$, we keep the part of $f(x,y,c)$ which does not depend on x (i.e. $y/(y+c)$) and we take the value of the rest (i.e. $x/((x+c)(x+y))$) at $c=0$ and $y=0$ (i.e. $1/x$). The product of these two factors (i.e. $y/(x(y+c))$) is called a counterterm and is subtracted from $f(x,y,c)$ to remove the logarithmic divergence for x. This procedure produces

$$f(x,y,c) - \frac{y}{x(y+c)} = -\frac{y}{y+c}\frac{xy+xc+yc}{x(x+c)(x+y)},$$

which is now convergent for the integral over x (it varies as λ^{-1} by power counting). If we make the same subtraction while fixing the variable x we obtain, subtracting both counterterms,

$$\begin{aligned} g(x,y,c) &= f(x,y,c) - \frac{y}{x(y+c)} - \frac{x}{y(x+c)} \\ &= -\frac{x^2y^2 + xy^3 + yx^3 + cy^3 + cx^3 + cyx^2 + cxy^2}{x(x+c)(x+y)(y+c)y}. \end{aligned}$$

This result is disappointing, because $g(x,y,c)$ is now linearly divergent if x is multiplied by λ, if y is multiplied by λ and if x and y are both multiplied by λ. In other words, $g(x,y,c)$ is still more divergent than $f(x,y,c)$. The miracle happens when we subtract the global linear divergence of $g(x,y,c)$. The final term,

$$\begin{aligned} \bar{f}(x,y,c) &= g(x,y,c) - g(x,y,0) - c\frac{\partial g(x,y,0)}{\partial c} \\ &= -c^2 \frac{xy+cx+cy}{x(x+c)(x+y)(y+c)y}, \end{aligned}$$

is now absolutely convergent for x, for y and for x, y.

2.2 The extension of distributions

From a mathematical point of view, renormalization theory can be considered as a method to extend a distribution to a larger domain.

The standard example is $1/x$. If $\phi(x)$ is a test function that vanishes at 0, then

$$\int_{-\infty}^{\infty} \mathrm{d}x \frac{\phi(x)}{x}$$

exists. The question is how it is possible to extend this distribution to general test functions. The existence of this extension is ensured by the Hahn–Banach theorem [21] and a formula for such extensions is

$$\int_{|x|<a} \mathrm{d}x \frac{\phi(x) - \phi(0)}{x} + \int_{|x|>a} \mathrm{d}x \frac{\phi(x)}{x},$$

for any positive parameter a. Hence various extensions are possible that are parametrized by a. Notice that the difference between two such integrals for a and a' is $(\log a' - \log a)\phi(0)$. Therefore, as distributions, two extensions of $1/x$ differ by $\log \Lambda \delta(x)$ for some Λ. The peculiarity of quantum field theory is that Λ can be determined by experiment.

The mathematical conditions for the existence of such an extension were investigated by Malgrange [22], Estrada [23] and Brunetti and Fredenhagen [35].

This extension method can be used to calculate, in some cases, the product of two distributions. For instance, by Fourier transform, it can be shown that $\delta(x-a)\delta(x) = \delta(a)$, for $a \neq 0$. However, if $a = 0$, the Fourier transform of the product diverges. This is exactly the same type of divergence as is met in the usual presentation of renormalization. The product of distributions $\delta(x)^2$ is zero for $x \neq 0$, thus a possible extension is $\delta(x)^2 = C\delta(x)$, where C is a constant determined by experiment.

In quantum field theory, causality, Poincaré invariance and unitarity were used by Stueckelberg and collaborators to provide a prescription to carry out this extension [24–27]. Bogoliubov and collaborators systematized this construction [18,28–30], which took its final form with Epstein and Glaser [31]. Nowadays, the extension method is called the "causal approach", and the case of QED is treated in detail in [32,33].

A (correct) proof of the validity of Bogoliubov's method was finally given by Hepp [19] in 1966 and by Zimmermann [34] in 1969.

Recently, the causal approach has been reinterpreted in terms of microlocal analysis [35]. This enabled these authors to provide the first renormalization of quantum field theory in curved spacetime. A pedagogical presentation of the causal approach can be found in [36].

From the causal point of view, the Feynman free propagators mix the plus and minus propagators in a too straightforward way. To circumvent the problem of divergence, each Feynman propagator must be split into its plus and minus parts. A similar point of view is used in Steinmann's axiomatic field theory of QED, recently reviewed in [37].

2.3 The product of distributions

The most radical approach to renormalization would be to define a product of distributions, which could lead to a nonlinear theory of distributions. Schwartz has shown that this is impossible in general [38], but the notion of a distribution can be extended to a more general kind of functions which can be multiplied. For a comparison with experimental results, we must project these new functions back onto standard distributions.

This approach was investigated by various authors [39–47]. The main drawback of these new generalized functions is that they lead to very heavy calculations. For instance, it is not difficult to show that [48]

$$\left(\frac{1}{x}\right)^2 = \frac{1}{x^2} + \pi^2 \delta(x)^2, \qquad (1)$$

but the computation of $(1/x)^3$ is already intractable. To understand the striking identity (1), we start from the continuous function $f(x) = x \log|x| - x$, which defines a distribution by $\int \mathrm{d}x f(x)\phi(x)$ for any test function $\phi(x)$. Then the distribution $1/x$ is defined as $\mathrm{d}^2 f/\mathrm{d}x^2$, the distribution $1/x^2$ is $-\mathrm{d}^3 f/\mathrm{d}x^3$, and $(1/x)^2$ is the product of the distribution $1/x$ with itself.

In spite of their complexity, these new generalized functions have found some applications in physics [49–52]. For instance, a definite value could be given to the curvature of a cone at its apex [50].

Notice that, as for the extension of distributions, microlocal analysis is of growing importance in the study of the new generalized functions [53].

3 Renormalization of QED

QED was renormalized to all orders by Dyson [15]. We can now interpret his prescriptions in the framework of the Schwinger equations. It is standard to define free, bare and renormalized propagators. The free electron Green function $S^0(q)$ is the Green function for an electron without electromagnetic interaction. The bare electron Green function $S(q)$ is the Green function for an electron with electromagnetic interaction, but without renormalization. In the perturbation expansion of $S(q)$, all terms (except the first one) are infinite. The renormalized Green function $\bar{S}(q)$ is the Green function for an electron with electromagnetic interaction, after renormalization. Similarly we define $D^0(q)$, $D(q)$ and $\bar{D}(q)$ as the free, bare and renormalized photon Green functions.[1]

3.1 The free propagators

The free electron propagator is

$$S^0(q) = (\gamma \cdot q - m + \mathrm{i}\epsilon)^{-1}.$$

The scalar product is defined by

$$\gamma \cdot q = \sum_{\lambda\mu} \gamma^\lambda g_{\lambda\mu} q^\mu,$$

where the pseudo-metric tensor $g_{\lambda\mu}$ is

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

[1] Strictly speaking, a propagator is a one-particle Green function, but in this paper propagator and Green function will be used indiscriminately. For simplicity, fermions (electrons + positrons) are called electrons.

All the electron propagators $S^0(q)$, $S(q)$ and $\bar{S}(q)$ are 4×4 complex matrix functions of the 4-vector q. If I is the 2×2 identity matrix and σ_x, σ_y, σ_z are the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the Dirac matrices can be written $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$,

$$\gamma^1 = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix},$$
$$\gamma^3 = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}.$$

The free photon propagator $D^0(q)$ is a complex 4×4 matrix with components $D^0_{\mu\nu}(q)$ defined by

$$D^0_{\mu\nu}(q) = -\frac{g_{\mu\nu}}{q^2+\mathrm{i}\epsilon} + (1-1/\xi)\frac{q_\mu q_\nu}{(q^2+\mathrm{i}\epsilon)^2}.$$

The term $1/\xi$ was introduced by Heisenberg to make $D^0(q)$ non-singular. The Green function used in classical electrodynamics is $D^{0\mathrm{T}}(q)$ defined as

$$D^{0\mathrm{T}}_{\mu\nu}(q) = -\frac{g_{\mu\nu}}{q^2+\mathrm{i}\epsilon} + \frac{q_\mu q_\nu}{(q^2+\mathrm{i}\epsilon)^2}.$$

A tensor $T_{\mu\nu}(q)$ such that $q^\mu T_{\mu\nu}(q) = 0$ is called transverse. It can be checked that $D^{0\mathrm{T}}_{\mu\nu}(q)$ is transverse, and non-singular in the space of transverse tensors.

Up to an eventual factor i, the expressions for the free Green functions $S^0(q)$ and $D^0_{\mu\nu}(q)$ are standard (see, e.g. [2], p. 93 and p. 36; [54], p. 184 and p. 190; [55], p. 218 and p. 253, for a complete description and a derivation).

3.2 The bare propagators

The Schwinger equations for bare electron and photon propagators were given by Bogoliubov and Shirkov [30] and transformed into the following integral equations in [11]:

$$\begin{aligned} S(q) &= S^0(q) \\ &+\mathrm{i}e_0^2 S^0(q)\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\gamma^\lambda D_{\lambda\lambda'}(p)\frac{\delta S(q-p)}{e_0\delta A^0_{\lambda'}(p)}, \quad (2)\\ D_{\mu\nu}(q) &= D^0_{\mu\nu}(q) \\ &-\mathrm{i}e_0^2 D^0_{\mu\lambda}(q)\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\mathrm{tr}\left[\gamma^\lambda \frac{\delta S(p)}{e_0\delta A^0_{\lambda'}(-q)}\right] D_{\lambda'\nu}(q), \quad (3)\end{aligned}$$

where $A^0_\lambda(p)$ is an external electromagnetic field and $\frac{\delta S(q)}{\delta A^0_\lambda(p)}$ is the functional derivative evaluated at $A^0_\lambda(p) = 0$.

The longitudinal part of the bare photon Green function is not modified by the interaction [2], and $D_{\lambda\mu}(q)$ can be written as the sum of its transverse and its longitudinal parts:

$$D_{\lambda\mu}(q) = D^{\mathrm{T}}_{\lambda\mu}(q) - \frac{1}{\xi}\frac{q_\lambda q_\mu}{(q^2+\mathrm{i}\epsilon)^2}, \quad (4)$$

where $D^{\mathrm{T}}_{\lambda\mu}(q)$ is transverse. In (3), the photon propagator $D_{\lambda'\nu}(q)$ is not integrated – it just multiplies the integral. This is not very convenient and it will be useful to introduce the bare vacuum polarization, denoted $\Pi_{\lambda\mu}(q)$ and defined by

$$[D^{-1}]_{\lambda\mu}(q) = (q_\lambda q_\mu - q^2 g_{\lambda\mu}) - \xi q_\lambda q_\mu - \Pi_{\lambda\mu}(q). \quad (5)$$

The vacuum polarization tensor $\Pi_{\lambda\mu}(q)$ is transverse [2]. If we multiply (5) by (4), we obtain

$$g_\lambda^{\ \nu} - \frac{q_\lambda q^\nu}{q^2} = \left((q_\lambda q_\mu - q^2 g_{\lambda\mu}) - \Pi_{\lambda\mu}(q)\right) D^{T\mu\nu}(q). \quad (6)$$

The left-hand side of (6) is the projector onto the transverse tensors.

We show in Sect. A.2 that

$$\Pi^{\lambda\mu}(q) = -\mathrm{i}e_0^2\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\mathrm{tr}\left[\gamma^\lambda \frac{\delta S(p)}{e_0\delta A^0_\mu(-q)}\right]. \quad (7)$$

3.3 Renormalized propagators

To obtain the Schwinger equations for the renormalized propagators, it is best is to start from the renormalized Lagrangian, and to follow the steps given by Bogoliubov and Shirkov [30], Itzykson and Zuber [2] or Rochev [56]. However, to give an idea of the result, we introduce some of Dyson's recipes.

The longitudinal part of the photon Green function is not modified by renormalization [2], and the renormalized photon propagator can be decomposed as

$$\bar{D}_{\lambda\mu}(q) = \bar{D}^{\mathrm{T}}_{\lambda\mu}(q) - \frac{1}{\xi}\frac{q_\lambda q_\mu}{(q^2+\mathrm{i}\epsilon)^2}, \quad (8)$$

where $\bar{D}^{\mathrm{T}}_{\lambda\mu}(q)$ is transverse.

Then, we introduce Dyson's relation between renormalized and bare Green functions ([2], p. 414):

$$\bar{S}(q)Z_2 = S(q), \quad (9)$$
$$Z_3\bar{D}^{\mathrm{T}}_{\mu\nu}(q) = D^{\mathrm{T}}_{\mu\nu}(q), \quad (10)$$
$$Z_3 e_0^2 = e^2, \quad (11)$$
$$m_0 = m + \delta m, \quad (12)$$

where Z_2 and Z_3 are (infinite) scalars independent of q, and e is the renormalized charge. Equation (11) was conjectured by Dyson [15] and proved by Ward [57]. Finally, the external field A^0_λ is renormalized as A_λ, so that

$$e_0 A^0_\lambda = e A_\lambda. \quad (13)$$

To introduce the mass renormalization, we must start from the differential form of the Schwinger equation for the bare electron propagator, where we reintroduce the external field, for later convenience,

$$\left[\mathrm{i}\gamma\cdot\partial - m_0 - e_0\gamma\cdot A^0(x)\right] S(x,y;A^0) = \delta(x-y) + \mathrm{i}e_0^2 \int \mathrm{d}^4 z \gamma^\mu D_{\mu\rho}(x,z;A^0) \frac{\delta S(x,y;A^0)}{e_0 \delta A^0_\rho(z)}. \quad (14)$$

Dyson showed that the relations (9)–(13), are valid in the presence of an external field. We use them in (14) to obtain

$$[\mathrm{i}\gamma\cdot\partial - m]\,\bar S(x,y;A) Z_2 = \delta(x-y) + \delta m \bar S(x,y;A) Z_2 + \mathrm{i}e^2 \int d^4 z \gamma^\mu \bar D_{\mu\rho}(x,z;A) \frac{\delta \bar S(x,y;A)}{e\delta A_\rho(z)} Z_2. \quad (15)$$

In (15), we have changed the gauge parameter ξ_0 of $D_{\mu\rho}(x,z;A^0)$ into $\xi = Z_3\xi_0$ (see [2], p. 414).

If we multiply (15) by

$$S^0(z,y;A) = [\mathrm{i}\gamma\cdot\partial - m - e\gamma\cdot A]^{-1} \quad (16)$$

and integrate over x we obtain the integral Schwinger equation for the renormalized electron propagator,

$$\bar S(x,y;A)Z_2 = S^0(x,y;A) + \mathrm{i}e^2 \int \mathrm{d}^4 z \mathrm{d}^4 z' S^0(x,z;A)\gamma^\lambda \bar D_{\lambda\lambda'}(z,z';A) \times \frac{\delta \bar S(z,y;A)}{e\delta A_{\lambda'}(z')} Z_2 + \delta m \int \mathrm{d}^4 z S^0(x,z;A) \bar S(z,y;A) Z_2. \quad (17)$$

In (17), we put $A = 0$ and we Fourier transform to find

$$\bar S(q) Z_2 = S^0(q) + \delta m S^0(q) \bar S(q) Z_2 + \mathrm{i}e^2 S^0(q) \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \gamma^\lambda \bar D_{\lambda\lambda'}(p) \frac{\delta \bar S(q-p)}{e\delta A_{\lambda'}(p)} Z_2. \quad (18)$$

This equation was given by Bogoliubov and Shirkov [30], as well as by Itzykson and Zuber ([2], p.481), except for the mass counterterm δm which was apparently overlooked by these authors. A complete derivation can be found in [56] (notice that there δm is our $Z_2\delta m$).

To obtain a convenient Schwinger equation for the renormalized photon propagator, we must introduce the renormalized vacuum polarization $\bar\Pi_{\lambda\mu}(q)$, defined by

$$[\bar D^{-1}]_{\lambda\mu}(q) = (q_\lambda q_\mu - q^2 g_{\lambda\mu}) - \xi q_\lambda q_\mu - \bar\Pi_{\lambda\mu}(q). \quad (19)$$

It may be useful to compare these definitions to those of Itzykson and Zuber [2]: $\bar D_{\mu\nu} = -\mathrm{i}\bar G_{\mu\nu}$, $\bar\Pi_{\mu\nu} = \mathrm{i}\bar\omega_{\mu\nu}$.

If we multiply (19) by (8), we obtain

$$g_\lambda^{\ \nu} - \frac{q_\lambda q^\nu}{q^2} = \left((q_\lambda q_\mu - q^2 g_{\lambda\mu}) - \bar\Pi_{\lambda\mu}(q)\right) \bar D^{T\mu\nu}(q). \quad (20)$$

If we compare (6) and (20), and use (10), we find

$$q_\lambda q_\mu - q^2 g_{\lambda\mu} - \bar\Pi_{\lambda\mu}(q) = Z_3(q_\lambda q_\mu - q^2 g_{\lambda\mu} - \Pi_{\lambda\mu}(q)). \quad (21)$$

Therefore, using (9) and (13)

$$\Pi^{\lambda\mu}(q) = -\mathrm{i}e_0^2 Z_2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{tr}\left[\gamma^\lambda \frac{\delta \bar S(p)}{e\delta A_\mu(-q)}\right].$$

Introducing this equation into (21), and using (11) we obtain

$$\bar\Pi_{\lambda\mu}(q) = (1 - Z_3)(q_\lambda q_\mu - q^2 g_{\lambda\mu}) - Z_2 \mathrm{i}e^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{tr}\left[\gamma_\lambda \frac{\delta \bar S(p)}{e\delta A^\mu(-q)}\right]. \quad (22)$$

Equations (18) and (22) will be the bases of recursive expressions for the renormalized electron and photon propagators.

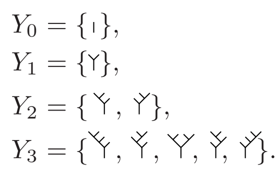
4 Tree expansion of propagators

For the convenience of the reader, and because the notation of [11] has been modified[2], we recall the description of photon and electron propagators in terms of planar binary trees. But first, we give a short introduction to planar binary trees.

4.1 Planar binary trees

A planar binary tree is a tree with a designated vertex called the root. To follow the notation of Loday and Ronco [58], we write the root vertex as ı. The other vertices are not explicitly drawn, but they are at the ends of each edge, which are \ or /. The trees are binary because each vertex has either zero or two children. They are planar because  is different from . The planar binary trees have an odd number of vertices and for each tree t we define $|t|$ as the integer such that t has $2|t|+1$ vertices. In other words, $|t|$ is the number of internal vertices. We call Y_n the set of planar binary trees t with such that $|t| = n$.

The planar binary trees with up to 7 vertices are



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[2] There are no longer trees with black or white roots. The color of the root is now indicated by the function φ itself. $\varphi(t;q)$ corresponds to a tree with a black root, $\varphi_{\mu\nu}(t;q)$ to a tree with a white root. This notation is more compact than the one of [11].

We denote by Y the set of all planar binary trees

$$Y = \bigcup_{n=0}^{\infty} Y_n .$$

Finally we consider the operation of *grafting* two trees, $\vee : Y_p \times Y_q \longrightarrow Y_{p+q+1}$, by which the roots of two trees t_1 and t_2 are joined into a new vertex that becomes the root of the tree $t = t_1 \vee t_2$, cf. [58]. For instance

$$\curlyvee \vee \curlyvee = \curlyvee\!\!\curlyvee . \tag{23}$$

It is clear that any tree t, except the 0-tree $|$, is the grafting of two uniquely determined trees t_l and t_r with orders $|t_l|, |t_r| \leq |t| - 1$.

4.2 Trees and propagators

The main trick of [11] was to write each propagator as a sum indexed by planar binary trees.

The bare electron Green function in Fourier space, $S(q)$ is written as a sum over planar binary trees t

$$S(q) = \sum_t e_0^{2|t|} \varphi^0(t;q). \tag{24}$$

Here e_0 is the bare electron charge (i.e. the electron charge before renormalization). The fact that the expansion is over e_0^2 (and not e_0) was justified in [11]. Similarly, the renormalized electron Green function is expanded over planar binary trees

$$\bar{S}(q) = \sum_t e^{2|t|} \bar{\varphi}^0(t;q). \tag{25}$$

In (25) e is the renormalized (finite) electron charge.

A word of caution is required here. The Schwinger equations (18) and (22), where the Ward identity $Z_1 = Z_2$ was used to eliminate Z_1 (see [2]), will be used to determine a recursive definition of $\bar{\varphi}^0(t;q)$. This definition does not ensure that $\bar{\varphi}^0(t;q)$ is finite. As explained below, $\bar{\varphi}^0(t;q)$ are sometimes divergent. Therefore, certain sums over t are required to obtain finite quantities. The expansion over trees is useful because the recursive relations for these sums are difficult to obtain directly, whereas they are immediate consequences of the recursive expressions for $\bar{\varphi}^0(t;q)$.

The bare and renormalized photon Green functions are written as

$$D_{\mu\nu}(q) = \sum_t e_0^{2|t|} \varphi^0_{\mu\nu}(t;q),$$

$$\bar{D}_{\mu\nu}(q) = \sum_t e^{2|t|} \bar{\varphi}^0_{\mu\nu}(t;q). \tag{26}$$

For the renormalization of the photon Green function and the vacuum polarization, it will be necessary to distinguish the photon Green function $D_{\mu\nu}(q)$ and the transverse photon Green function $D^{\mathrm{T}}_{\mu\nu}(q)$. Since all terms $\varphi^0_{\mu\nu}(t;q)$ and $\bar{\varphi}^0_{\mu\nu}(t;q)$ are transverse for $t \neq |$, the transverse renormalized propagator is

$$\bar{D}^{\mathrm{T}}_{\mu\nu}(q) = \varphi^{\mathrm{T}}_{\mu\nu}(|;q) + \sum_{|t|>0} e^{2|t|} \bar{\varphi}^0_{\mu\nu}(t;q),$$

where $\varphi^{\mathrm{T}}_{\mu\nu}(|;q) = D^{0\mathrm{T}}_{\mu\nu}(q)$.

The bare and renormalized vacuum polarization are expanded similarly:

$$\Pi_{\lambda\mu}(q) = \sum_{|t|>0} e_0^{2|t|} \psi^0_{\lambda\mu}(t;q), \tag{27}$$

$$\bar{\Pi}_{\lambda\mu}(q) = \sum_{|t|>0} e^{2|t|} \bar{\psi}^0_{\lambda\mu}(t;q). \tag{28}$$

For later convenience, we finally define

$$\psi^0_{\lambda\mu}(|;q) = q_\lambda q_\mu - q^2 g_{\lambda\mu},$$

so that

$$[D^{0^{-1}}]_{\lambda\mu}(q) = (q_\lambda q_\mu - q^2 g_{\lambda\mu}) - \xi q_\lambda q_\mu \tag{29}$$

$$= \psi^0_{\lambda\mu}(|;q) - \xi q_\lambda q_\mu . \tag{30}$$

A few identities will be useful in the sequel:

$$D^0_{\lambda\mu}(q)(q^\mu q_\nu - q^2 g^\mu{}_\nu) = -q^2 D^{0\mathrm{T}}_{\lambda\nu}(q),$$
$$\psi^0_{\lambda\lambda'}(|;q) D^{0\lambda'\mu'}(q) \psi^0_{\mu'\mu}(|;q) = \psi^0_{\lambda\mu}(|;q),$$
$$D^0_{\lambda\lambda'}(q) \psi^{0\lambda'\mu'}(|;q) D^0_{\mu'\mu}(q) = D^{0\mathrm{T}}_{\lambda\mu}(q).$$

Notice that $-q^2 D^{0\mathrm{T}}_{\lambda\nu}(q)$ is the projector onto the transverse tensors.

The tree representation of the photon propagator enjoys the following property [11]

$$\varphi^0_{\mu\nu}(t_l \vee t_r;q) = \varphi^0_{\mu\lambda}(| \vee t_r;q)[(D^0)^{-1}]^{\lambda\lambda'}(q) \times \varphi^0_{\lambda'\nu}(t_l;q).$$

This equality is non-trivial only if $t_l \neq |$. But then $\varphi^0_{\mu\lambda}(| \vee t_r;q)$ and $\varphi^0_{\lambda'\nu}(t_l;q)$ are transverse [11]; this cancels the term proportional to ξ and we can write

$$\varphi^0_{\mu\nu}(t_l \vee t_r;q) = \varphi^0_{\mu\lambda}(| \vee t_r;q)\psi^{0\lambda\lambda'}(|;q)\varphi^0_{\lambda'\nu}(t_l;q). \tag{31}$$

The fact that $\varphi^0_{\mu\nu}(t_l \vee t_r;q)$ is a product for $t_l \neq |$ can be checked in Appendix 3.

As shown in Sect. A.2, $\psi^0_{\mu\nu}(t_l \vee t_r) = 0$ if $t_l \neq |$ and

$$\psi^0_{\mu\nu}(| \vee t_r;q) = \psi^0_{\mu\lambda}(|;q)\varphi^{0\lambda\lambda'}(| \vee t_r;q)\psi^0_{\lambda'\nu}(|;q). \tag{32}$$

Hence, the vacuum polarizations are finally expanded as

$$\Pi_{\lambda\mu}(q) = \sum_t e_0^{2|t|+2} \psi^0_{\lambda\mu}(| \vee t;q), \tag{33}$$

$$\bar{\Pi}_{\lambda\mu}(q) = \sum_t e^{2|t|+2} \bar{\psi}^0_{\lambda\mu}(| \vee t;q). \tag{34}$$

4.3 Trees and renormalization constants

According to Dyson's multiplicative renormalization of QED, we consider three renormalization constants Z_2, Z_3 and δm. We expand these constants over planar binary trees:

$$Z_2 = 1 + \sum_{|t|>0} e^{2|t|}\zeta_2(t), \quad \text{with} \quad \zeta_2(\shortmid) = 1, \qquad (35)$$

$$Z_3 = 1 - \sum_{|t|>0} e^{2|t|}\zeta_3(t), \quad \text{with} \quad \zeta_3(\shortmid) = 1, \qquad (36)$$

$$\delta m = \sum_{|t|>0} e^{2|t|}\zeta_m(t), \quad \text{with} \quad \zeta_m(\shortmid) = 0. \qquad (37)$$

The minus sign in the definition of Z_3 follows [10] and will be explained in Sect. 8.3. The values of $\zeta_2(t)$, $\zeta_3(t)$ and $\zeta_m(t)$ cannot be determined for each tree. They will be summed over sets of trees, and the sum will be determined by the renormalization conditions (see Sect. 8.2).

4.4 Recursive equations for bare propagators

In [11], we have obtained recursive relations for $\varphi(t)$ and $\varphi_{\lambda\mu}(t)$.

For the electron propagator, $\varphi^0(t;q)$, for $t = t_l \vee t_r$ can be obtained from $\varphi^0_{\lambda\mu}(t_l;q)$ and $\varphi^1(t_r;q;\lambda,p)$ by the equation

$$\varphi^0(t;q) = \mathrm{i}S^0(q)\int \frac{\mathrm{d}^4p}{(2\pi)^4}\gamma^\lambda\varphi^0_{\lambda\mu}(t_l;p) \times \varphi^1(t_r;q-p;\mu,p). \qquad (38)$$

The higher components $\varphi^n(t;q;\{\lambda,p\}_{1,n})$ for $n>0$ are defined recursively by

$$\begin{aligned}\varphi^n(t;q;\{\lambda,p\}_{1,n}) &= S^0(q)\gamma^{\lambda_1}\varphi^{n-1}(t;q+p_1;\{\lambda,p\}_{2,n})\\ &+\mathrm{i}\sum_{k=0}^{n}\int\frac{\mathrm{d}^4p}{(2\pi)^4}S^0(q)\gamma^\lambda\varphi^k_{\lambda\lambda'}(t_l;p;\{\lambda,p\}_{1,k})\\ &\times\varphi_\Sigma^{n-k+1}(t_r;q-p;\lambda',p+P_k,\{\lambda,p\}_{k+1,n}),\end{aligned}$$

where we have denoted $P_k = p_1+\cdots+p_k$, $(P_0=0)$ and $\{\lambda,p\}_{1,n} = \lambda_1,p_1,\ldots,\lambda_n,p_n$. The initial data are

$$\begin{aligned}\varphi^0(\shortmid;q) &= S^0(q),\\ \varphi^1(\shortmid;q;\lambda_1,p_1) &= S^0(q)\gamma^{\lambda_1}S^0(q+p_1),\\ \varphi^n(\shortmid;q;\{\lambda,p\}_{1,n}) &= S^0(q)\gamma^{\lambda_1}S^0(q+p_1)\gamma^{\lambda_2}\cdots\gamma^{\lambda_n}\\ &\times S^0(q+p_1+\cdots+p_n). \qquad (39)\end{aligned}$$

The symbol $\varphi_\Sigma^{n+1}(t;q;z,\{z\}_{1,n})$ is defined as the sum of $n+1$ terms, where the first variable $z=(\lambda,p)$ is exchanged in turn with each of the variables $z_i=(\lambda_i,p_i)$:

$$\begin{aligned}\varphi_\Sigma^{n+1}(t;q;z,\{z\}_{1,n}) &= \varphi^{n+1}(t;q;z,\{z\}_{1,n})\\ &+\sum_{k=1}^{n-1}\varphi^{n+1}(t;q;\{z\}_{1,k},z,\{z\}_{k+1,n})\\ &+\varphi^{n+1}(t;q;\{z\}_{1,n},z).\end{aligned}$$

Similarly, for the photon propagator, $\varphi^0_{\mu\nu}(t;q)$ is obtained from the equation

$$\varphi^0_{\mu\nu}(t;q) = -\mathrm{i}D^0_{\mu\lambda}(q)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\left[\gamma^\lambda\varphi^1(t_r;p;\lambda',-q)\right] \times\varphi^0_{\lambda'\nu}(t_l;q). \qquad (40)$$

The higher components $\varphi^n_{\mu\nu}(t;q;\{\lambda,p\}_{1,n})$ satisfy the recursive relation

$$\begin{aligned}\varphi^n_{\mu\nu}(t;q;\{\lambda,p\}_{1,n}) &= -\mathrm{i}\sum_{k=0}^{n}\int\frac{\mathrm{d}^4p}{(2\pi)^4}D^0_{\mu\lambda}(q)\\ &\times\mathrm{tr}\left[\gamma^\lambda\varphi_\Sigma^{k+1}(t_r;p;\lambda',-q-P_k,\{\lambda,p\}_{1,k})\right]\\ &\times\varphi^{n-k}_{\lambda'\nu}(t_l;q+P_k;\{\lambda,p\}_{k+1,n}),\end{aligned}$$

with the initial data

$$\begin{aligned}\varphi^0_{\mu\nu}(\shortmid;q) &= D^0_{\mu\nu}(q),\\ \varphi^n_{\mu\nu}(\shortmid;q;\{\lambda,p\}_{1,n}) &= 0 \quad\text{for}\quad n\geq 1.\end{aligned}$$

4.5 The pruning operator

In this section, we introduce the pruning operator P which will prove very useful to obtain a recursive expression for renormalized propagators. If t is a tree, $P(t)$ is a sum of $n(t)$ terms of the form $u_j\otimes v_j$, where u_j and v_j are planar binary trees. More formally

$$P(t) = \sum_{j=1}^{n(t)} u_j\otimes v_j. \qquad (41)$$

Before we fully define $P(t)$, we want to show why it is useful. If, for each tree t, $\varphi(t)$ and $\psi(t)$ are 4×4 complex matrices, we call the convolution of φ and ψ the quantity

$$(\varphi\bar{\star}\psi)(t) = \sum_{i=1}^{n(t)}\varphi(u_i)\psi(v_i).$$

The main property of this convolution was established in [11]. If

$$X(\lambda) = \sum_t \lambda^{|t|}x(t) \quad\text{and}\quad Y(\lambda) = \sum_t\lambda^{|t|}y(t),$$

with $x(\shortmid) = y(\shortmid) = 0$, then

$$X(\lambda)Y(\lambda) = \sum_t\lambda^{|t|}(x\bar{\star}y)(t).$$

In other words, the pruning operator and the convolution enable us to multiply series indexed by planar binary trees.

This nice property justifies the trouble of introducing $P(t)$. First, $n(t)$, the number of terms in (41), is defined by $n(\shortmid)=0$ and

$$\begin{aligned}n(t) &= 0 \quad\text{if}\quad t = t_l\vee\shortmid,\\ n(t) &= n(t_r)+1 \quad\text{if}\quad t=t_l\vee t_r,\quad t_r\neq\shortmid. \qquad (42)\end{aligned}$$

Finally, $P(t)$ is determined recursively by $P(|) = 0$ and

$$P(t) = 0 \quad \text{if} \quad t = t_l \vee |,$$

$$P(t) = (t_l \vee |) \otimes t_r + \sum_{j=1}^{n(t_r)} (t_l \vee u_j) \otimes v_j \quad \text{if} \quad t = t_l \vee t_r, \quad t_r \neq |. \tag{43}$$

The trees u_j and v_j in (43) are generated by (41) for $t = t_r$. For instance

$$P(|) = P(|\vee|) = P((|\vee|)\vee|) = P(((|\vee|)\vee|)\vee|) = P((|\vee(|\vee|))\vee|) = 0,$$

$$P(|\vee(|\vee|)) = (|\vee|) \otimes (|\vee|), \qquad P((|\vee|)\vee(|\vee|)) = ((|\vee|)\vee|) \otimes (|\vee|),$$

$$P(|\vee((|\vee|)\vee|)) = (|\vee|) \otimes ((|\vee|)\vee|), \qquad P(|\vee(|\vee(|\vee|))) = (|\vee|) \otimes (|\vee(|\vee|)) + (|\vee(|\vee|)) \otimes (|\vee|).$$

We show in the appendix that the pruning operator is co-associative, that is

$$(P \otimes \mathrm{id}) \otimes P = (\mathrm{id} \otimes P) \otimes P. \tag{44}$$

Therefore the convolution is associative.

We consider on trees the structure of an associative algebra $T(Y)$ given by the (non-commutative) tensor product, $T(Y) = \mathbb{C} \oplus Y \oplus Y^{\otimes 2} \oplus \ldots$, and we set the root $|$ as the unit: $| \otimes t = t \otimes | = t$. Then we extend P to $T(Y)$ as a multiplicative map, but P does not preserve the unit, since $P(|)$ is not equal to $| \otimes |$. We can define a coproduct

$$\Delta^P t = | \otimes t + P(t) + t \otimes |,$$
$$\Delta^P | = | \otimes |.$$

This Δ^P is the coproduct of a Hopf algebra over planar binary trees. Its antipode is given by the recursive formula

$$S_P(t) = -t - (\mathrm{Id} \bar{\star} S_P)(t) = -t - (S_P \bar{\star} \mathrm{Id})(t), \tag{45}$$

for $t \neq |$, and $S_P(|) = |$.

To define the convolution of $x(t)$ and $y(t)$, we needed the condition $x(|) = y(|) = 0$. When this condition is not satisfied, we have two solutions. The first solution is to isolate the root, so that

$$\begin{aligned} X(\lambda)Y(\lambda) &= x(|)y(|) + (X(\lambda) - x(|))y(|) \\ &\quad + x(|)(Y(\lambda) - y(|)) \\ &\quad + (X(\lambda) - x(|))(Y(\lambda) - y(|)) \\ &= x(|)y(|) + x(|) \sum_{|t|>0} \lambda^{|t|} y(t) \\ &\quad + \sum_{|t|>0} \lambda^{|t|} x(t) y(|) + \sum_{|t|>0} \lambda^{|t|} (x \bar{\star} y)(t). \end{aligned}$$

The second solution is to use the coproduct Δ^P. Thus, we define the convolution $\star$ by

$$(x \star y)(t) = x(|)y(t) + (x \bar{\star} y)(t) + x(t)y(|),$$
$$(x \star y)(|) = x(|)y(|).$$

With this alternative convolution, the equality

$$X(\lambda)Y(\lambda) = \sum_t \lambda^{|t|} (x \star y)(t) \tag{46}$$

is satisfied even if x or y is not zero on the root.

In our final formulas, we prefer to use the convolution $\bar{\star}$ because its ensures the recursivity of the expressions (the trees in $(x \bar{\star} y)(t)$ are strictly smaller than t).

A last point concerning notation. If φ and ψ depend on other arguments, we leave them inside φ and ψ. For example

$$(\varphi(q) \bar{\star} \psi(q))(t) = \sum_{j=1}^{n(t)} \varphi(u_j; q) \psi(v_j; q).$$

4.6 The self-energy

As a first application of the convolution defined in the previous section, we introduce the tree expansion for the electron self-energy.

The bare electron self-energy $\Sigma(q)$ is defined by

$$S^{-1}(q) = \gamma \cdot q - m - \Sigma(q) = \sum_t e_0^{2|t|} \psi^0(t; q),$$

where

$$\psi^0(|; q) = \gamma \cdot q - m \quad \text{and} \quad \Sigma(q) = -\sum_{|t|>0} e_0^{2|t|} \psi^0(t; q),$$

so that $\psi^0(|; q)\varphi^0(|; q) = 1$. The pruning operator is used to define the expansion of the bare self-energy over trees in terms of the expansion of the bare electron Green function over trees:

$$\psi^0(t) = -\psi^0(|)\varphi^0(t)\psi^0(|) - \psi^0(|)(\varphi^0 \bar{\star} \psi^0)(t), \tag{47}$$

$$\varphi^0(t) = -\varphi^0(|)\psi^0(t)\varphi^0(|) - (\varphi^0 \bar{\star} \psi^0)(t)\varphi^0(|). \tag{48}$$

In terms of the antipode, (47) can be rewritten

$$\psi^0(t) = \psi^0(|; q)(\varphi^0(q) \circ S_P)(t)\psi^0(|; q). \tag{49}$$

Similarly, for the renormalized self-energy, we have

$$\bar{\psi}^0(t) = \psi^0(|; q)(\bar{\varphi}^0(q) \circ S_P)(t)\psi^0(|; q). \tag{50}$$

We must give some details concerning the meaning of expressions like $(\varphi^0(q) \circ S_P)(t)$. Because of its definition (45), the antipode S_P acting on t generates a sum of products of trees. The action of $\varphi^0(q)$ on this sum is linearly extended from its action on Y to $T(Y)$. In other words

$$\varphi^0(t_1 + t_2; q) = \varphi^0(t_1; q) + \varphi^0(t_2; q),$$
$$\varphi^0(\lambda t; q) = \lambda \varphi^0(t; q).$$

For a product of trees, we do not want to simply multiply two Feynman diagrams for the electron propagator; we must cancel one of the free propagators among them. Thus, the product is

$$\varphi^0(t_1 t_2; q) = \varphi^0(t_1; q)\psi^0(|; q)\varphi^0(t_2; q). \tag{51}$$

This operation becomes clear if one tries it on some examples given in Appendix 3. Since the (matrix) product on the right-hand side of (51) is not commutative, the algebra product on trees is not commutative either.

In the presence of an external field A^0, the definition (16) of $S^0(z,y;A^0)$ gives, after inversion and Fourier transform,

$$\psi^0(\mathsf{I};q;A^0) = \gamma\cdot q - m - e_0\gamma\cdot A^0(q). \tag{52}$$

Thus we obtain at $A^0 = 0$

$$\psi^0(\mathsf{I};q) = \gamma\cdot q - m,$$
$$\psi^1(\mathsf{I};q;\lambda,p) = -\gamma^\lambda,$$
$$\psi^n(\mathsf{I};q;\{\lambda,p\}_{1,n}) = 0, \quad \text{for} \quad n>1.$$

The components of $\psi^0(t;q)$ for the other trees t are obtained by using the chain rule for the functional derivative of (47) with respect to $e_0 A^0_{\lambda_i}(p_i)$, taken at $A^0 = 0$. For $n = 1$, this gives the same result as in Sect. 6.4 of [11].

Finally it is shown in Sect. A.3 that the bare self-energy can be calculated from the recursive equation

$$\psi^0(t;q) = \mathrm{i}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\gamma^\lambda \varphi^0_{\lambda\lambda'}(t_l;p) g(t_r;q-p;\lambda',p), \tag{53}$$

where

$$g(t_r;q-p;\lambda',p) = -(\varphi^1(q-p;\lambda',p)\star\psi^0(q))(t_r)$$
$$= (\varphi^0(q)\star\psi^1(q-p;\lambda',p))(t_r).$$

It can be shown that $\psi^0(t;q)$ is a sum of one-particle irreducible (1PI) Feynman diagrams. The proof is recursive: if $\psi^0(t;q)$ is such a sum for all $|t|<n$, then $\psi^1(t;q-p;\lambda',p)$ is also a sum of 1PI diagrams, with one external photon line. Therefore, according to its second definition, $g(t_r;q-p;\lambda',p)$ is a sum of diagrams made up of an electron propagator (on the left) followed by a reduced vertex (on the right). According to (53), $\psi^0(t;q)$ is obtained by plugging the photon propagator $\varphi^0_{\lambda\lambda'}(t_l;p)$ on the external photon line of $g(t_r;q-p;\lambda',p)$, and closing this photon propagator over the electron propagator of $g(t_r;q-p;\lambda',p)$. This gives again a sum of 1PI diagrams.

4.7 The higher components $\varphi^n(t)$

For a complete recursive solution of the renormalized propagators, we must define the higher components $\bar\varphi^n(t)$ and $\bar\varphi^n_{\mu\nu}(t)$. As for the bare propagators (see [11]), they are defined as the functional derivative with respect to an external electromagnetic field.

Let us be more accurate concerning this external field. As noticed by Bogoliubov and Shirkov, (17) is not the Schwinger equation for QED with an external electromagnetic field, since the latter involves tadpole diagrams which are absent from (17). However, (17) is the Schwinger equation for a renormalizable theory (i.e. QED without tadpoles), and Dyson's relations (9)–(13) still hold.

In the real space, the bare and renormalized electron Green functions are expanded as

$$S(x,y;A) = \sum_t e_0^{2|t|}\varphi^0(t;x,y;A),$$
$$\bar S(x,y;A) = \sum_t e^{2|t|}\bar\varphi^0(t;x,y;A).$$

In these expressions, we do not distinguish between A and A^0 because A^0 comes always multiplied by e_0 and $e_0 A^0 = eA$. On the root, $\varphi^0(\mathsf{I};x,y;A) = S^0(x,y;A)$. The higher components of $\varphi^0(t;x,y;A)$ and $\bar\varphi^0(t;x,y;A)$ must satisfy

$$\frac{\delta}{e\delta A_\lambda(z)}\varphi^n(t;x,y;\{\lambda,z\}_{1,n};A)$$
$$= \varphi^{n+1}_\Sigma(t;x,y;\lambda,z,\{\lambda,z\}_{1,n};A),$$

where the notation φ^{n+1}_Σ and $\{\lambda,z\}_{1,n}$ is defined in Sect. 4.4.

Since our purpose is QED without external field, A is just used to take functional derivatives, and the higher components we actually need are

$$\varphi^n(t;x,y;\{\lambda,z\}_{1,n}) = \varphi^n(t;x,y;\{\lambda,z\}_{1,n};A)$$

for $A = 0$.

At $A = 0$, the theory becomes translationally invariant, and a Fourier transform gives us

$$\frac{\delta}{e\delta A_\lambda(p)}\varphi^n(t;q;\{\lambda,p\}_{1,n}) = \varphi^{n+1}_\Sigma(t;q;\lambda,p,\{\lambda,p\}_{1,n}).$$

In the recursive equations for $\varphi(t)$ we meet products of propagators such as $\varphi^0(t_1;q)\varphi^0(t_2;q)$. In the real space, this gives a space-time convolution of $\varphi^0(t_1;x,y)$ and $\varphi^0(t_2;x,y)$. We take the functional derivative of this convolution with respect to $A(z)$, we Fourier transform the result and we obtain at $A = 0$

$$\frac{\delta}{e\delta A_\lambda(p)}\left(\varphi^0(t_1;q)\varphi^0(t_2;q)\right)$$
$$= \varphi^0(t_1;q)\varphi^1(t_2;q;\lambda,p)$$
$$+\varphi^1(t_1;q;\lambda,p)\varphi^0(t_2;q+p).$$

This expression satisfies energy-momentum conservation.

To take the functional derivatives of the recursive equations for renormalized quantities, we need the independence of the renormalization constants with respect to the external field. There are various ways to prove this. For instance, the differential form of the Ward identity ((21) in [11]) is

$$\frac{\partial\varphi^n(t;q;\{\lambda,p\}_{1,n})}{\partial q_\mu} = -\varphi^{n+1}_\Sigma(t;q;\mu,0,\{\lambda,p\}_{1,n}).$$

The Ward identity is also valid for the renormalized electron propagator [59], so that

$$\frac{\partial\bar\varphi^n(t;q;\{\lambda,p\}_{1,n})}{\partial q_\mu} = -\bar\varphi^{n+1}_\Sigma(t;q;\mu,0,\{\lambda,p\}_{1,n}).$$

From the definitions (49) and (50) of the bare and renormalized self-energies we obtain

$$\frac{\partial\psi^n(t;q;\{\lambda,p\}_{1,n})}{\partial q_\mu} = -\psi_\Sigma^{n+1}(t;q;\mu,0,\{\lambda,p\}_{1,n}),$$
$$\frac{\partial\bar\psi^n(t;q;\{\lambda,p\}_{1,n})}{\partial q_\mu} = -\bar\psi_\Sigma^{n+1}(t;q;\mu,0,\{\lambda,p\}_{1,n}).$$

Now we start from the relation between the renormalized and bare self-energies, for instance

$$\bar\psi^0(\Upsilon;q) = \psi^0(\Upsilon;q) + \zeta_2(\Upsilon)(\gamma\cdot q - m) - \zeta_m(\Upsilon). \tag{54}$$

On the one hand, we take the derivative of (54) with respect to q_μ and use the Ward identities (and the fact that $\zeta_2(\Upsilon)$ and $\zeta_m(\Upsilon)$ do not depend on q) to obtain

$$-\bar\psi^1(\Upsilon;q;\mu,0) = -\psi^1(\Upsilon;q;\mu,0) + \zeta_2(\Upsilon)\gamma^\mu. \tag{55}$$

On the other hand, we take the functional derivative of (54) at $A = 0$ and we obtain

$$\bar\psi^1(\Upsilon;q;\mu,p) = \psi^1(\Upsilon;q;\mu,p) - \zeta_2(\Upsilon)\gamma^\mu + \zeta_2'(\Upsilon)(\gamma\cdot q - m) - \zeta_m'(\Upsilon), \tag{56}$$

where $\zeta_2'(\Upsilon)$ and $\zeta_m'(\Upsilon)$ denote the derivative of $\zeta_2(\Upsilon)$ and $\zeta_m(\Upsilon)$ with respect to $A_\mu(p)$ at $A = 0$. The term $\zeta_2(\Upsilon)\gamma^\mu$ comes from the functional derivative of (52).

If we take the value $p = 0$ in (56) and compare with (55) we obtain $\zeta_2'(\Upsilon) = 0$ and $\zeta_m'(\Upsilon) = 0$. Further differentiation shows that $\zeta_2^{(n)}(\Upsilon) = 0$ and $\zeta_m^{(n)}(\Upsilon) = 0$. We can apply this proof to any tree t, once the subdivergences have been subtracted. A similar proof can be given for $\zeta_3(t)$, using Furry's theorem instead of the Ward identities. This proof assumes that the renormalization conditions do not depend on A (e.g. minimal subtraction).

More physically, $S(x,y;A)$ has the same singular structure at $x = y$ as $S^0(x,y)$, except for logarithmic terms that are integrable. Thus, the renormalization constants are determined by $S^0(x,y)$.

This method of functional derivatives avoids the usual renormalization of vertex diagrams. Renormalizing propagators is sufficient.

5 The renormalized electron propagator

In this section, we show that the recursive equation for the electron propagator is

$$\begin{aligned}\bar\varphi^0(t;q) &= \rho(t)\varphi^0(|;q) + \zeta_m(t)\varphi^0(|;q)^2 \\ &+\varphi^0(|;q)(\zeta_m\bar\star\bar\varphi^0(q))(t) \\ &+\mathrm{i}\varphi^0(|;q)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\gamma^\lambda\bar\varphi^0_{\lambda\lambda'}(t_l;p)\bar\varphi^1(t_r;q-p;\lambda',p),\end{aligned} \tag{57}$$

where $\rho(t) = -\zeta_2(t) - (\rho\bar\star\zeta_2)(t) = \zeta_2\circ S_P(t)$, starting at $\rho(\Upsilon) = -\zeta_2(\Upsilon)$.

It is natural to define a new quantity $\alpha(t;q)$ by $\alpha(t;q) = 0$ for $t = |$ and, for $t = t_l \vee t_r$,

$$\alpha(t;q) = \mathrm{i}e^2S^0(q) \times \int\frac{\mathrm{d}^4p}{(2\pi)^4}\gamma^\lambda\bar\varphi_{\lambda\lambda'}(t_l;p)\bar\varphi^1(t_r;q-p;\lambda',p).$$

Then we consider (18) and in the integral over p, we expand the photon propagator over trees t_l using (26) and the electron propagator over trees t_r using (25). We recognize a sum of $\alpha(t_l\vee t_r)$ and the integral becomes

$$\mathrm{i}e^2S^0(q)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\gamma^\lambda\bar D_{\lambda\lambda'}(p)\frac{\delta\bar S(q-p)}{e\delta A_{\lambda'}(p)} = \sum_t e^{2|t|}\alpha(t;q).$$

In the other terms of (18), we expand $\bar S(q)$ and renormalization constants over trees using (25), (35), (36) and (37). We replace products by convolutions $\star$ according to (46) in all expressions except the integral and we obtain

$$\begin{aligned}\sum_t e^{2|t|}(\bar\varphi^0(q)\star\zeta_2)(t) &= \varphi^0(|;q) \\ &+\sum_t e^{2|t|}(\alpha(q)\star\zeta_2)(t) \\ &+\sum_t e^{2|t|}\varphi^0(|;q)(\zeta_m\star\bar\varphi^0(q)\star\zeta_2)(t).\end{aligned}$$

The bold step is now to identify the terms corresponding to a given tree t. This yields

$$\begin{aligned}(\bar\varphi^0(q)\star\zeta_2)(t) &= \varphi^0(|;q)\epsilon(t) + (\alpha(q)\star\zeta_2)(t) \\ &+\varphi^0(|;q)(\zeta_m\star\bar\varphi^0(q)\star\zeta_2)(t),\end{aligned} \tag{58}$$

where $\epsilon(t) = 1$ if $t = |$ and $\epsilon(t) = 0$ otherwise. In fact, this step is a bit too bold. If (58) is summed over t, then the result determines a finite renormalized electron propagator. However, the fact that the sum is finite does not mean that each term is finite, and it turns out that all the $\bar\varphi^0(t;q)$ are not finite. Thus, we shall have to use a summation over certain classes of trees to get finite results.

To simplify expression (58), we follow Kreimer [60] and compute $(\bar\varphi^0(q)\star\zeta_2\star\zeta_2\circ S_P)(t)$. The basic property of the antipode is $\mathrm{Id}\star S_P = |\epsilon$; therefore

$$\zeta_2\star\zeta_2\circ S_P = \zeta_2(\mathrm{Id}\star S_P) = \zeta_2(|)\epsilon = \epsilon. \tag{59}$$

The associativity of $\star$ is crucial here. On the one hand, $\big(\bar\varphi^0(q)\star(\zeta_2\star\zeta_2\circ S_P)\big)(t) = \bar\varphi^0(q)(t)$ according to (59). On the other hand, we calculate $\big((\bar\varphi^0(q)\star\ \zeta_2)\star\ \zeta_2\circ\ S_P\big)$, where we replace $\bar\varphi^0(q)\star\zeta_2$ by the right-hand side of (58). Equation (59) gives us

$$\begin{aligned}\big((\bar\varphi^0(q)\star\zeta_2)\star\zeta_2\circ S_P\big) &= \zeta_2\circ S_P(t)\varphi^0(|;q) \\ &+\varphi^0(|;q)(\zeta_m\star\bar\varphi^0(q))(t) + \alpha(t).\end{aligned}$$

From the associativity of $\star$ and the definition of $\alpha(t;q)$ we obtain our final recursive equation (57) for the electron propagator.

The recursive equation is completed by the equation for the higher components of $\bar\varphi^0(t;q)$. If we take the functional derivative of (57), and make the same simplification as in [11]), we obtain

$$\begin{aligned}
&\bar\varphi^n(t;q;\{\lambda,p\}_{1,n}) = \varphi^0(|;q)\gamma^{\lambda_1}\bar\varphi^{n-1}(t;q;\{\lambda,p\}_{2,n})\\
&\quad-\rho(t)\varphi^0(|;q)\delta_{n,0} + \zeta_m(t)\varphi^0(|;q)\bar\varphi^n(|;q;\{\lambda,p\}_{1,n})\\
&\quad+\varphi^0(|;q)(\zeta_m\bar\star\bar\varphi^n(q;\{\lambda,p\}_{1,n}))(t)\\
&\quad+\mathrm{i}\varphi^0(|;q)\sum_{k=0}^{n}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\gamma^\lambda\bar\varphi^k_{\lambda\lambda'}(t_l;p;\{\lambda,p\}_{1,k})\\
&\quad\times\bar\varphi^{n-k+1}_{\Sigma}(t_r;q-p;\lambda',p,\{\lambda,p\}_{k+1,n}). \qquad (60)
\end{aligned}$$

Another useful formula can be obtained by defining

$$\begin{aligned}
\bar f^0(t;q) &= (\bar\varphi^0(q)\star\zeta_2)(t) \qquad (61)\\
&= \bar\varphi^0(t;q) + (\bar\varphi^0(q)\bar\star\zeta_2)(t)\\
&\quad+\varphi^0(|;q)\zeta_2(t),\\
\bar f^1(t;q;\lambda',p) &= (\bar\varphi^1(q;\lambda',p)\star\zeta_2)(t). \qquad (62)
\end{aligned}$$

With this notation (58) is rewritten

$$\begin{aligned}
&\bar f^0(t;q) = \zeta_m(t)\varphi^0(|;q)^2 + \varphi^0(|;q)(\zeta_m\bar\star\bar f^0(q))(t)\\
&\quad+\mathrm{i}\varphi^0(|;q)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\gamma^\lambda\bar\varphi^0_{\lambda\lambda'}(t_l;p)\\
&\quad\times\bar f^1(t_r;q-p;\lambda',p). \qquad (63)
\end{aligned}$$

The higher components $\bar f^n(t;q)$ are obtained by functional derivative of (63), as explained in Sect. 4.7. The recursive equation for $\bar f^n(t;q)$ is the same as (60), where φ is replaced by f and the term $\rho(t)\varphi^0(|;q)\delta_{n,0}$ is suppressed.

6 The renormalized photon propagator

Bogoliubov and Shirkov [30] have shown that the renormalization of two Feynman diagrams linked by a single photon (or electron) line is obtained by an independent renormalization of each of the two subgraphs. In our language, this means that the renormalized form of (31) is

$$\bar\varphi^0_{\mu\nu}(t_l\vee t_r;q) = \bar\varphi^0_{\mu\lambda}(|\vee t_r;q)\psi^{0\lambda\lambda'}(|;q)\bar\varphi^0_{\lambda'\nu}(t_l;q). \qquad (64)$$

Therefore, all trees for the photon propagator can be renormalized once we have renormalized the special trees $|\vee t_r$. Now we show that the recursive equation for the renormalized photon term $\bar\varphi^0_{\mu\lambda}(|\vee t_r;q)$ is

$$\begin{aligned}
\bar\varphi^0_{\mu\nu}(|\vee t;q) &= \zeta_3(|\vee t)\varphi^{\mathrm{T}}_{\mu\nu}(|;q)\\
&\quad-\mathrm{i}\varphi^{\mathrm{T}}_{\mu\lambda}(|;q)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\left[\gamma^\lambda\bar f^1(t;p;\lambda',-q)\right]\\
&\quad\times\varphi^{\mathrm{T}}_{\lambda'\nu}(|;q), \qquad (65)
\end{aligned}$$

where $\bar f^1$ was defined in the previous section.

To prove this, we start from several remarks: we have $\bar\psi^0_{\mu\nu}(t_l\vee t_r)=0$ is $t_l\neq|$ and

$$\bar\psi^0_{\mu\nu}(|\vee t_r;q) = \psi^0_{\mu\lambda}(|;q)\bar\varphi^{0\lambda\lambda'}(|\vee t_r;q)\psi^0_{\lambda'\nu}(|;q).$$

Because of this close analogy between photon propagator and vacuum polarization, we shall rewrite (22) as

$$\begin{aligned}
&\sum_t e^{2|t|}\bar\varphi^0_{\mu\nu}(|\vee t;q) = \varphi^{\mathrm{T}}_{\mu\lambda}(|;q)\bar\Pi^{\lambda\lambda'}(q)\varphi^{\mathrm{T}}_{\lambda'\nu}(|;q)\\
&\quad= (1-Z_3)\varphi^{\mathrm{T}}_{\lambda\mu}(|;q) - Z_2\mathrm{i}e^2\varphi^{\mathrm{T}}_{\mu\lambda}(|;q)\\
&\quad\times\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\left[\gamma^\lambda\frac{\delta\bar S(p)}{e\delta A_{\lambda'}(-q)}\right]\varphi^{\mathrm{T}}_{\lambda'\nu}(|;q).
\end{aligned}$$

We rewrite this expression to isolate the root components:

$$\begin{aligned}
&\sum_t e^{2|t|}\bar\varphi^0_{\mu\nu}(|\vee t;q) = (1-Z_3)\varphi^T_{\lambda\mu}(|;q)\\
&\quad-\mathrm{i}e^2\varphi^{\mathrm{T}}_{\mu\lambda}(|;q)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\left[\gamma^\lambda\frac{\delta\bar S(p)}{e\delta A_{\lambda'}(-q)}\right]\varphi^{\mathrm{T}}_{\lambda'\nu}(|;q)\\
&\quad-\mathrm{i}e^2\varphi^{\mathrm{T}}_{\mu\lambda}(|;q)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\left[\gamma^\lambda\frac{\delta\bar S(p)}{e\delta A_{\lambda'}(-q)}\right](Z_2-1)\\
&\quad\times\varphi^{\mathrm{T}}_{\lambda'\nu}(|;q).
\end{aligned}$$

We expand all quantities over trees, using

$$\frac{\delta\bar S(p)}{e\delta A_{\lambda'}(-q)} = \sum_t e^{2|t|}\bar\varphi^1(t;p;\lambda',-q),$$

and we multiply through the pruning operator. Then we identify the terms corresponding to the same tree and we obtain

$$\begin{aligned}
&\bar\varphi^0_{\mu\nu}(|\vee t;q) = \zeta_3(|\vee t)\varphi^{\mathrm{T}}_{\mu\nu}(|;q) + \zeta_2(t)\varphi^0_{\mu\nu}(\Upsilon;q)\\
&\quad-\mathrm{i}\varphi^{\mathrm{T}}_{\mu\lambda}(|;q)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\left[\gamma^\lambda\bar\varphi^1(t;p;\lambda',-q)\right]\\
&\quad\times\varphi^{\mathrm{T}}_{\lambda'\nu}(|;q)\\
&\quad-\mathrm{i}\varphi^{\mathrm{T}}_{\mu\lambda}(|;q)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\left[\gamma^\lambda(\bar\varphi^1(p;\lambda',-q)\bar\star\zeta_2)(t)\right]\\
&\quad\times\varphi^{\mathrm{T}}_{\lambda'\nu}(|;q).
\end{aligned}$$

From the definition (62) for $\bar f^1$, we can rewrite this expression as our recursive equation (65).

The higher components are obtained very simply by taking the functional derivative of (65). Since $\varphi^{\mathrm{T}}_{\mu\nu}(|;q)$ is independent of the external field, we obtain

$$\begin{aligned}
&\bar\varphi^n_{\mu\nu}(|\vee t;q;\{\lambda,p\}_{1,n}) = -\mathrm{i}\varphi^{\mathrm{T}}_{\mu\lambda}(|;q)\\
&\quad\times\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\left[\gamma^\lambda\bar f^{n+1}_{\Sigma}(t;p;\lambda',-q,\{\lambda,p\}_{1,n})\right]\\
&\quad\times\varphi^{\mathrm{T}}_{\lambda'\nu}(|;q).
\end{aligned}$$

For the other trees, we use

$$\begin{aligned}
&\bar\varphi^n_{\mu\nu}(t_l\vee t_r;q;\{\lambda,p\}_{1,n}) = \sum_{k=0}^{n}\bar\varphi^k_{\mu\lambda}(|\vee t_r;q;\{\lambda,p\}_{1,k})\\
&\quad\times\psi^{0\lambda\lambda'}(|;q)\bar\varphi^{n-k}_{\lambda'\nu}(t_l;q;\{\lambda,p\}_{k+1,n}).
\end{aligned}$$

6.1 Properties of renormalized photon propagator

From (63) and (65) we can deduce that the renormalized photon propagator does not depend on any $\zeta_2(t)$. In fact, we shall prove that $\bar{f}^0(t;q)$, $\bar{f}^1(t;q;\lambda,p)$ and $\bar{\varphi}^0_{\mu\nu}(t;q)$ do not depend on any $\zeta_2(t')$. To do this, we reintroduce a non-zero external field A. The property is clearly true for $t = |$. If it is true for all trees with $|t| < N$, let us take a tree with $|t| = N$. Because of (63), $\bar{f}^0(t;q)$ does not depend on any $\zeta_2(t')$. Since $\bar{f}^1(t;q;\lambda,p)$ is obtained by a functional derivative of $\bar{f}^0(t;q)$ with respect to eA, it does not depend on any $\zeta_2(t')$ either (eA does not depend on any $\zeta_2(t')$). If $t = |\vee t_r$, because of (65), $\bar{\varphi}^0_{\mu\nu}(|\vee t_r;q)$ does not depend on any $\zeta_2(t')$ since none of the terms on the right hand side do. Finally, if t is not of the form $t = |\vee t_r$, it is of the form $t = t_l \vee t_r$, and $\bar{\varphi}^0_{\mu\nu}(t_l \vee t_r;q)$ is obtained from $\bar{\varphi}^0_{\mu\nu}(|\vee t_r;q)$ and $\bar{\varphi}^0_{\mu\nu}(t_l;q)$, which do not depend on any $\zeta_2(t')$.

With the same reasoning, we see that $\bar{f}^0(t;q)$ and $\bar{\varphi}^0_{\mu\nu}(t;q)$ are independent of the gauge parameter ξ for $t \neq |$.

7 Electron self-energy

To calculate the electron self-energy, we start from (57) that we rewrite

$$\bar{\varphi}^0(t;q) = \rho(t)\varphi^0(|;q) + \zeta_m(t)\varphi^0(|;q)^2 + \alpha(t;q) + \varphi^0(|;q)(\zeta_m \bar{\star} \bar{\varphi}^0(q))(t).$$

The self-energy is obtained by introducing the last equation into (47). This gives us

$$\begin{aligned}\bar{\psi}^0(t;q) &= -\rho(t)\psi^0(|;q) - \zeta_m(t) \\ &-\psi^0(|;q)\alpha(t;q)\psi^0(|;q) \\ &-(\rho\bar{\star}\bar{\psi}^0(q))(t) - (\zeta_m\bar{\star}\bar{\varphi}^0(q))(t)\psi^0(|;q) \\ &-\varphi^0(|;q)(\zeta_m\bar{\star}\bar{\psi}^0(q))(t) - (\zeta_m\bar{\star}\bar{\varphi}^0(q)\bar{\star}\bar{\psi}^0(q))(t) \\ &-\psi^0(|;q)(\alpha(q)\bar{\star}\bar{\psi}^0(q))(t).\end{aligned}$$

If we factorize ζ_m and use (48), the expression reduces to

$$\begin{aligned}\bar{\psi}^0(t;q) = &-\rho(t)\psi^0(|;q) - \zeta_m(t) \\ &- \psi^0(|;q)\alpha(t;q)\psi^0(|;q) - (\rho\bar{\star}\bar{\psi}^0(q))(t) \\ &- \psi^0(|;q)(\alpha(q)\bar{\star}\bar{\psi}^0(q))(t).\end{aligned}$$

From the definition of $\alpha(t;q)$ and the result of Sect. A.3 we obtain

$$\begin{aligned}\bar{\psi}^0(t;q) = &-\rho(t)\psi^0(|;q) - \zeta_m(t) - (\rho\bar{\star}\bar{\psi}^0(q))(t) \\ &+ \mathrm{i}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\gamma^\lambda \bar{\varphi}^0_{\lambda\lambda'}(t_l;p)\bar{g}(t_r;q-p;\lambda',p),\end{aligned}$$

where

$$\bar{g}(t_r;q-p;\lambda',p) = -(\bar{\varphi}^1(q-p;\lambda',p) \star \bar{\psi}^0(q))(t_r) \tag{66}$$

$$= (\bar{\varphi}^0(q) \star \bar{\psi}^1(q-p;\lambda',p))(t_r). \tag{67}$$

It can be shown that the only term proportional to $\psi^0(|;q)$ in $-\rho(t)\psi^0(|;q) - (\rho\bar{\star}\bar{\psi}^0(q))(t)$ is $\zeta_2(t)\psi^0(|;q)$.

8 Renormalization and Ward identities

In this section, we describe in what sense the renormalization of Feynman diagrams or expansion over trees are incompatible with the Ward identity $Z_1 = Z_2$. To do that, we first need the Connes–Kreimer Hopf algebra of Feynman diagrams for QED.

8.1 Connes–Kreimer algebra for QED

The Hopf algebra of renormalization for QED is obtained by adapting the results of Connes and Kreimer [9]. According to standard results, QED is renormalized once three types of 1PI diagrams are renormalized: the self-energy, the vacuum polarization and the reduced vertex Feynman diagrams [15,2].

Letting Γ be one of these three diagrams, the Connes–Kreimer coproduct of Γ is written

$$\begin{aligned}\Delta_{CK}\Gamma = &\,\Gamma\otimes 1 + 1\otimes\Gamma \\ &+ \sum_{\{\gamma i,\gamma' i',\dots\}} \gamma_{(i)}\gamma'_{(i')}\cdots\otimes\Gamma/\{\gamma_{(i)},\gamma'_{(i')},\dots\}.\end{aligned}$$

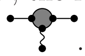
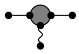
In this expression, the sum runs over all sets of disjoint renormalization parts of Γ. A renormalization part γ of Γ is a 1PI subgraph of Γ, different from Γ itself, such that γ has two or three (amputated) external lines. Two renormalization parts γ and γ' are disjoint if they have no vertex in common. There are three types of renormalization parts: self-energy 1PI diagrams ($\gamma =$), vacuum polarization 1PI diagrams ($\gamma =$) and reduced vertex 1PI diagrams ($\gamma =$). The index (i) depends on the type of the renormalization part. If $\gamma =$ then i is 0 or 2, if $\gamma =$ then i is 3, if $\gamma =$ then i is 1. There are three differences between our notation and that of Connes and Kreimer [9]. Firstly, we use the notation $\{\gamma i,\gamma' i',\dots\}$ for what they denote by γ. This enables us to refer only to 1PI diagrams. Secondly our two-lines vertices are $\gamma\cdot p - m$ and m, whereas they use $\gamma\cdot p$ and m. With this change, we can write the renormalized propagators of massive QED as a sum of counterterms multiplied by derivatives of the bare propagators with respect to the mass m. Thirdly, the indices of their two-line vertices are 0 and 1 and ours are 0, 1, 2, 3. This last change is justified by the identification of the counterterms with the renormalization factors δm, Z_1, Z_2 and Z_3.

Finally let us define $\Gamma/\{\gamma_{(i)},\gamma'_{(i')},\dots\}$. We start by the definition of $\Gamma/\gamma_{(i)}$. It varies with the type of renormalization part and with the index (i).

If $\gamma =$ is a self-energy 1PI diagram, then i can be 0 and 2. In the complete diagram Γ, the self-energy subdiagram $\gamma =$ is a part of an electron propagator . In the term $\Gamma/\gamma_{(0)}$, the propagator diagram is replaced by a product of two free propagators , in the term $\Gamma/\gamma_{(2)}$ the propagator diagram is replaced by a single free propagator .

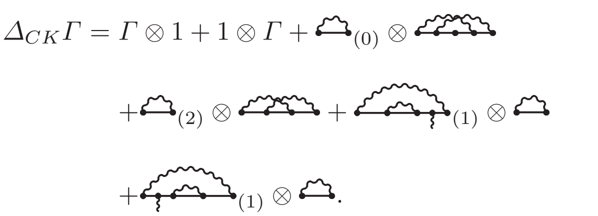
If $\gamma =$ is a vacuum polarization 1PI diagram, then i is 3. In the complete diagram Γ, the vacuum polarization

$\gamma =$ is a part of a photon propagator . In the term $\Gamma/\gamma_{(3)}$, the propagator diagram is replaced by a single free propagator .

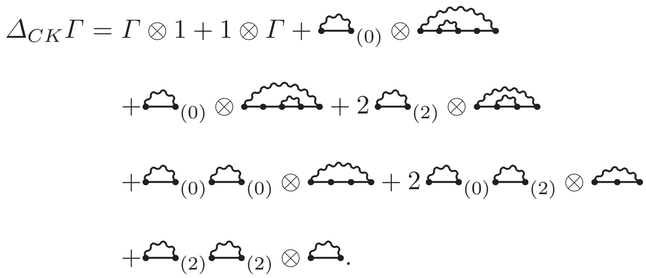
If $\gamma =$ is a reduced vertex 1PI diagram, then i is 1. In the complete diagram Γ, the reduced vertex $\gamma =$ is a part of a vertex diagram . In the term $\Gamma/\gamma_{(1)}$, the vertex diagram  is replaced by a free vertex .

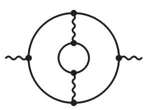
The terms $\Gamma/\{\gamma_{(i)}, \gamma'_{(i')}, \dots\}$ are then defined recursively. For instance, to define $\Gamma'' = \Gamma/\{\gamma_{(i)}, \gamma'_{(i')}\}$, we first put $\Gamma' = \Gamma/\gamma_{(i)}$, so that $\Gamma'' = \Gamma'/\gamma'_{(i')}$.

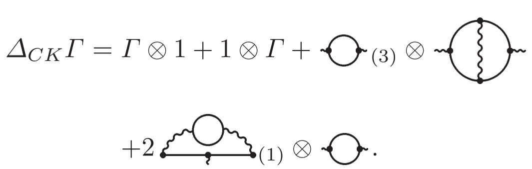
A few examples might be useful to the reader. For Γ =  the coproduct is



For Γ=  the coproduct is



For $\Gamma =$  the coproduct is



The renormalization of a self-energy, a vacuum polarization or a reduced vertex diagram proceeds in two steps. In the first step, we assume that all the counterterms $C_i(\gamma)$ of the divergent subdiagrams γ of Γ have been determined. Then the subdivergences of Γ are removed by the following formulas:

$$\bar{R}(\Gamma; q) = U(\Gamma; q) + \sum_{\{\gamma i, \gamma' i', \dots\}} C_i(\gamma) C_{i'}(\gamma') \cdots \times U(\Gamma/\{\gamma_{(i)}, \gamma'_{(i')}, \dots\}; q)$$

for a self-energy or a vacuum polarization diagram Γ and

$$\bar{R}(\Gamma; q; \lambda, p) = U(\Gamma; q; \lambda, p) + \sum_{\{\gamma i, \gamma' i', \dots\}} C_i(\gamma) C_{i'}(\gamma') \cdots \times U(\Gamma/\{\gamma_{(i)}, \gamma'_{(i')}, \dots\}; q; \lambda, p)$$

for a reduced vertex diagram Γ.

In these expressions $U(\Gamma; \cdot)$ is the value of the Feynman diagram Γ and $U(\Gamma/\{\gamma_{(i)}, \gamma'_{(i')}, \dots\}; \cdot)$ is the value of the Feynman diagram $\Gamma/\{\gamma_{(i)}, \gamma'_{(i')}, \dots\}$. The dot $\cdot$ represents the arguments of U (i.e. "q" or "$q; \lambda, p$").

In the second step, we determine the counterterms $C_i(\Gamma)$ of the divergent graph Γ. Again, we must distinguish three cases. A self-energy diagram is linearly divergent; thus we must remove two terms. From Lorentz covariance, we can write the renormalized value of the diagram Γ as

$$R(\Gamma; q) = \bar{R}(\Gamma; q) + C_0(\Gamma) + C_2(\Gamma)(\gamma \cdot q - m). \tag{68}$$

The factor $\gamma \cdot q - m$ in the second counterterm of any self-energy diagram γ explains why the product of two free propagators present in $\Gamma/\gamma_{(0)}$ becomes a single propagator in $\Gamma/\gamma_{(2)}$: the counterterm cancels out one of the two propagators.

A vacuum polarization diagram is quadratically divergent; thus we should have to remove three terms. However, Lorentz covariance and the Ward identities for the photon propagators cancel the first two counterterms, and the renormalized value of Γ is

$$R(\Gamma; q) = \bar{R}(\Gamma; q) + C_3(\Gamma) \psi^0_{\lambda\mu}(\shortmid; q). \tag{69}$$

There is a subtlety here. We know that $\bar{R}(\Gamma; q)$ is transverse. Thus it is necessary to use a transverse counterterm $C_3(\Gamma)\psi^0_{\lambda\mu}(\shortmid; q)$ to obtain a transverse vacuum polarization $\bar{\Pi}_{\lambda\mu}(q)$. However, to obtain a single free photon propagator in the operation $/\Gamma_{(3)}$, we used a counterterm of the form $C_3(\Gamma) D^{0^{-1}}(q)$, which cancels out one of the product of two free photon propagators that would normally be present for a quadratically divergent diagram (this product disappears only because of Lorentz covariance and the Ward identities). Therefore, the renormalization rules define actually

$$\begin{aligned} R'(\Gamma; q) &= R(\Gamma; q) - C_3(\Gamma) \xi q_\lambda q_\mu \\ &= \bar{R}(\Gamma; q) + C_3(\Gamma) D^{0^{-1}}(q). \end{aligned}$$

In the Landau gauge $\xi = 0$ the two renormalizations are identical. In another gauge we obtain

$$\begin{aligned} \bar{\Pi}'_{\lambda\mu}(q) &= \sum_{\Gamma=} e^{|\Gamma|} R'(\Gamma; q) \\ &= \bar{\Pi}_{\lambda\mu}(q) + (Z_3 - 1)\xi q_\lambda q_\mu, \end{aligned} \tag{70}$$

where $|\Gamma|$ is the number of vertices of Γ. The corresponding renormalized propagator is

$$\begin{aligned} \bar{D}'_{\lambda\mu}(q) &= (D^0(q)^{-1} - \bar{\Pi}'(q))^{-1} \\ &= \bar{D}^{\mathrm{T}}_{\lambda\mu}(q) - \frac{1}{Z_3 \xi} \frac{q_\lambda q_\mu}{(q^2 + \mathrm{i}\epsilon)^2}. \end{aligned}$$

The bare gauge parameter is multiplied by Z_3, as expected from [2], p. 414. The dependence on ξ of the propagators and renormalization factors is known explicitly to all orders (see [61] for a recent review).

Finally, a reduced vertex diagram is logarithmically divergent, and its renormalized value is

$$R(\Gamma;q;\lambda,p) = \bar{R}(\Gamma;q;\lambda,p) + C_1(\Gamma). \tag{71}$$

The infinite constants $C_i(\Gamma)$ are determined from the renormalization conditions.

8.2 Renormalization conditions

The value of $C_0(\Gamma)$, $C_1(\Gamma)$, $C_2(\Gamma)$ and $C_3(\Gamma)$ are determined from the renormalization conditions. For instance, the conditions used by Zimmermann are

$$R(\Gamma;0) = \bar{R}(\Gamma;0) - C_2(\Gamma)m + C_0(\Gamma) = 0,$$
$$\frac{\partial R(\Gamma;0)}{\partial q_\mu} = \frac{\partial \bar{R}(\Gamma;0)}{\partial q_\mu} + C_2(\Gamma)\gamma^\mu = 0.$$

Since $\mathrm{tr}[\gamma_\mu\gamma^\mu] = 16$, we can define $C_2(\Gamma)$ by

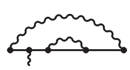
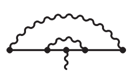
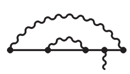
$$C_2(\Gamma) = -\frac{1}{16}\mathrm{tr}\left[\frac{\partial \bar{R}(\Gamma;0)}{\partial q_\mu}\gamma_\mu\right],$$

so that $C_0(\Gamma) = C_2(\Gamma)m - \bar{R}(\Gamma;0)$. Similar conditions determine $C_3(\Gamma)$ for a vacuum polarization diagram. Many other renormalization conditions are possible, such as minimal subtraction or mass shell renormalization [2,12].

8.3 Ward identities and counterterms

Now, if we take a self-energy diagram Γ and enumerate the electron lines of Γ from 1 to n, we can consider the reduced vertex diagrams Γ_j (for $j = 1,\ldots,n$) obtained by inserting a photon line in the jth electron line of Γ.

For instance, if $\Gamma=$ , then $n = 3$ and

$\Gamma_1 =$ , $\Gamma_2 =$ , $\Gamma_3 =$ .

The bare and renormalized value of Γ_j will be denoted $U(\Gamma_j;q;\lambda,p)$ and $R(\Gamma_j;q;\lambda,p)$. If the electron line j belongs to an electron loop, then $U(\Gamma_j;q;\lambda,p) = 0$ and $R(\Gamma_j;q;\lambda,p) = 0$ by Furry's theorem. According to the Ward identity ([62] p. 243) for bare and renormalized values

$$\frac{\partial U(\Gamma;q)}{\partial q_\mu} = -\sum_j U(\Gamma_j;q;\mu,0),$$
$$\frac{\partial R(\Gamma;q)}{\partial q_\mu} = -\sum_j R(\Gamma_j;q;\mu,0).$$

Moreover, it can be shown that

$$\frac{\partial \bar{R}(\Gamma;q)}{\partial q_\mu} = -\sum_j \bar{R}(\Gamma_j;q;\mu,0).$$

Therefore, using (71), combined with the derivative of (68) with respect to q_μ and the Ward identities, we obtain

$$C_2(\Gamma) = -\sum_j C_1(\Gamma_j). \tag{72}$$

When we renormalize a given diagram Γ, we often have a reduced vertex subdiagram γ_j corresponding to a self-energy diagram γ (in other words, γ_j is obtained by branching a photon line on the jth electron line of γ). But, generally, all the γ_j (obtained by branching a photon line on all the electron lines of γ) are not subdiagrams of Γ. Because of this, we cannot use (72), and the cancellations coming from (72) are obtained when all the Feynman diagrams are summed. This is what we mean when we say that the renormalization of a Feynman diagram is not compatible with $Z_1 = Z_2$.

Consider for instance the example of the renormalization of  treated in Sect. 8.1. Among the counter terms, we have $\Gamma_{1(1)}$ and $\Gamma_{3(1)}$ as factors of $\otimes$, but not $\Gamma_{2(1)}$. Because of this, we cannot use (72) to eliminate the counterterms $C_1(\Gamma_i)$ in the renormalization of Γ. When the C_1 counterterms cannot be eliminated, we say that the renormalization scheme is not compatible with $Z_1 = Z_2$.

The trees represent a sum of Feynman diagrams, and we first thought that the renormalization of trees was compatible with $Z_1 = Z_2$. This is true for all trees up to order 3 (i.e. to e^6). However, we discovered a tree of order 6 (e^{12}) which is not compatible with $Z_1 = Z_2$, confirming the reputation of perversity of the renormalization theory.

In the next sections, we shall give sums of Feynman diagrams which are compatible with $Z_1 = Z_2$ (i.e. all C_1 counterterms are eliminated). But first, we need the relation between the renormalization factors and the counterterms. This relation was given by Dyson [15] and in the early books on quantum field theory (e.g. [63]). In our notation, this relation is

$$Z_1 = 1 + \sum_{\Gamma_j=} e^{|\Gamma_j|-1}C_1(\Gamma_j), \tag{73}$$


$$Z_2 = 1 - \sum_{\Gamma=} e^{|\Gamma|}C_2(\Gamma), \tag{74}$$


$$Z_3 = 1 - \sum_{\Gamma=} e^{|\Gamma|}C_3(\Gamma), \tag{75}$$


$$Z_2\delta m = \sum_{\Gamma=} e^{|\Gamma|}C_0(\Gamma). \tag{76}$$


We saw that the each tree t can be considered as a sum of Feynman diagrams. Let $t(\Gamma)$ be the unique tree where

$\varGamma$ appears. According to our definitions of the self-energy and vacuum polarization, we have for the self-energy

$$t = -\sum_{t(\varGamma)=t} \varGamma,$$

where $\varGamma$ are self-energy diagrams and for the vacuum polarization

$$t = \sum_{t(\varGamma)=t} \varGamma,$$

where $\varGamma$ are vacuum polarization diagrams. Therefore, the relation between renormalization factors and trees is

$$Z_2 = 1 + \sum_{|t|>0} e^{2|t|}\zeta_2(t),$$

$$\delta m = \sum_{|t|>0} e^{2|t|}\zeta_m(t),$$

$$Z_3 = 1 - \sum_{|t|>0} e^{2|t|}\zeta_3(t).$$

The relation between ζ_i and counterterms is

$$\zeta_2(t) = -\sum_{t(\varGamma)=t} C_2(\varGamma),$$

$$\zeta_3(t) = \sum_{t(\varGamma)=t} C_3(\varGamma),$$

$$\zeta_m(t) = \sum_{t(\varGamma)=t} C_0(\varGamma) - (\zeta_m \bar{\star} \zeta_2)(t).$$

9 Total number of loops

The simplest way to obtain finite results from our expansion over trees is to sum over all trees of order ℓ, i.e. over all diagrams with 2ℓ vertices. For instance, we can define

$$\varphi^0_{\lambda\mu}(\ell;q) = \sum_{|t|=\ell} \varphi^0_{\lambda\mu}(t;q),$$

$$\psi^0_{\lambda\mu}(\ell;q) = \sum_{|t|=\ell} \psi^0_{\lambda\mu}(t;q),$$

$$\varphi^0(\ell;q) = \sum_{|t|=\ell} \varphi^0(t;q),$$

$$\psi^0(\ell;q) = \sum_{|t|=\ell} \psi^0(t;q),$$

with similar definitions for the renormalized quantities. According to this definition, the photon propagator, vacuum polarization, electron propagator and self-energy are

$$D_{\lambda\mu}(q) = D^0_{\lambda\mu}(q) + \sum_{\ell>0} e_0^{2\ell}\varphi^0_{\lambda\mu}(\ell;q),$$

$$\varPi_{\lambda\mu}(q) = \sum_{\ell>0} e_0^{2\ell}\psi^0_{\lambda\mu}(\ell;q),$$

$$S(q) = S^0(q) + \sum_{\ell>0} e_0^{2\ell}\varphi^0(\ell;q),$$

$$\varSigma(q) = -\sum_{\ell>0} e_0^{2\ell}\psi^0(\ell;q),$$

with similar relations for the renormalized quantities.

The sum of all trees of order ℓ is also a sum of all Feynman diagrams with ℓ loops. For instance, $\psi^0(\ell,q)$ is the sum of all the self-energy diagrams with 2ℓ vertices. It was proved that QED is renormalizable order by order [2], and that it is compatible with the identity $Z_1 = Z_2$. In other words, the Dyson equation $Z_2\bar{S}(q;e) = S(q;e_0/(Z_3^{1/2}))$ gives $\bar{S}(q;e)$ as a series over e, where each term is finite. Since this identity is the only thing that we used to define our renormalization, $\bar{\varphi}^0(\ell;q)$ is the term of order 2ℓ in the series defining $\bar{S}(q;e)$. Therefore, $\bar{\varphi}^0(\ell;q)$ is finite, and the same is true for $\bar{\psi}^0(\ell;q)$, $\bar{\varphi}^0_{\lambda\mu}(\ell;q)$ and $\bar{\psi}^0_{\lambda\mu}(\ell;q)$.

The results that were obtained for trees can now be translated into the loop order approach by summing over all trees of a given order. For instance, the pruning operator becomes

$$P(\ell) = \sum_{|t|=\ell} P(t) = \sum_{m=1}^{\ell-1} m \otimes (\ell - m).$$

Notice that this coproduct is the dual of the product of series. Thus, it is co-commutative. From this definition we obtain the reduced and full convolutions

$$(\varphi \bar{\star} \psi)(\ell) = \sum_{m=1}^{\ell-1} \varphi(m) \otimes \psi(\ell - m),$$

$$(\varphi \star \psi)(\ell) = \sum_{m=0}^{\ell} \varphi(m) \otimes \psi(\ell - m).$$

The antipode becomes $S_P(0) = 0$ and

$$S_P(\ell) = -\ell - \sum_{m=1}^{\ell-1} S_P(m)(\ell - m).$$

We also define

$$Z_{2,\ell} = \sum_{|t|=\ell} \zeta_2(t),$$

$$Z_{3,\ell} = \sum_{|t|=\ell} \zeta_3(t),$$

$$Z_{m,\ell} = \sum_{|t|=\ell} \zeta_m(t),$$

so that $Z_2 = \sum_\ell e^{2\ell} Z_{2,\ell}$, etc. Notice that $Z_{2,0} = Z_{3,0} = 1$ and $Z_{m,0} = 0$.

9.1 Recursive relations

By summing the recursive equations for planar binary trees, we obtain recursive equations for the 2ℓth order terms of the bare and renormalized propagators. These recursive equations decrease the combinatorial explosion of Feynman diagrams. The number of integrals to carry out at ℓ loops increases with the exponential of ℓ if all Feynman diagrams are calculated, whereas it increases as a polynomial in ℓ with the recursive expressions.

The case of massless QED is obtained from the recursive equations by setting $m = 0$ and all $Z_{m,\ell} = 0$.

Using the notation of Sect. 4.4 we can write a recursive relation for the electron propagator.

$$\begin{aligned}\varphi^n(\ell; q; \{\lambda, p\}_{1,n}) &= S^0(q)\gamma^{\lambda_1}\varphi^{n-1}(\ell; q + p_1; \{\lambda, p\}_{2,n}) \\ &+\mathrm{i}\sum_{m=0}^{\ell-1}\sum_{k=0}^{n}\int \frac{\mathrm{d}^4 p}{(2\pi)^4} S^0(q)\gamma^{\lambda}\varphi^{k}_{\lambda\lambda'}(m; p; \{\lambda, p\}_{1,k}) \\ &\times\varphi_{\Sigma}^{n-k+1}(\ell - m - 1; q - p; \lambda', p + P_k, \{\lambda, p\}_{k+1,n}),\end{aligned} \tag{77}$$

The initial data for $\ell = 0$ are $\varphi^n(0; q) = \varphi^n(\shortmid; q)$ and $\varphi^n_{\lambda\mu}(0; q) = \varphi^n_{\lambda\mu}(\shortmid; q)$.

For the photon propagator,

$$\begin{aligned}\varphi^n_{\mu\nu}(\ell; q; \{\lambda, p\}_{1,n}) &= -\mathrm{i}\sum_{m=0}^{\ell-1}\sum_{k=0}^{n}\int \frac{\mathrm{d}^4 p}{(2\pi)^4} D^0_{\mu\lambda}(q) \\ &\times\mathrm{tr}\left[\gamma^{\lambda}\varphi_{\Sigma}^{k+1}(m; p; \lambda', -q - P_k, \{\lambda, p\}_{1,k})\right] \\ &\times\varphi^{n-k}_{\lambda'\nu}(\ell - m - 1; q + P_k; \{\lambda, p\}_{k+1,n}).\end{aligned} \tag{78}$$

For the vacuum polarization, we have similarly

$$\begin{aligned}\psi^{n\mu\nu}(\ell; q; \{\lambda, p\}_{1,n}) &= -\mathrm{i}\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \\ &\times\mathrm{tr}\left[\gamma^{\mu}\varphi_{\Sigma}^{n+1}(\ell - 1; p; \nu, -q - P_n, \{\lambda, p\}_{1,n})\right].\end{aligned} \tag{79}$$

The same kind of relations can be obtained for the renormalized quantities. We list here just a few of them for the renormalized electron propagator:

$$\begin{aligned}\bar\varphi^0(\ell; q) &= S_P(Z_{2,\ell})S^0(q) + \sum_{p=1}^{\ell} Z_{m,p}S^0(q)\bar\varphi^0(\ell - p; q) \\ &+ \mathrm{i}\sum_{m=0}^{\ell-1}\sum_{k=0}^{n}\int \frac{\mathrm{d}^4 p}{(2\pi)^4} S^0(q)\gamma^{\lambda} \\ &\times\bar\varphi^0_{\lambda\lambda'}(m; p)\bar\varphi^1(\ell - m - 1; q - p; \lambda', p).\end{aligned}$$

and the renormalized vacuum polarization

$$\begin{aligned}\bar\psi^0_{\mu\nu}(\ell; q) &= Z_{3,\ell}\psi^0_{\mu\nu}(0; q) \\ &-\mathrm{i}\sum_{m=0}^{\ell-1}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\mathrm{tr}\left[\gamma^{\mu}\bar\varphi^1(\ell - m - 1; p; \nu, -q)\right] Z_{2,m}.\end{aligned}$$

In the case of massless QED, the Hopf algebra of renormalization at the loop order will be studied in detail in a forthcoming publication [64].

10 Number of electron and photon loops

In the previous section, we showed that renormalization and $Z_1 = Z_2$ are compatible if all the diagrams with a given number of loops are summed. In the present section, we see that this property still holds if we sum all diagrams having a given number of electron and photon loops.

The expansion over photon and electron loops is not as compact as that over the total number of loops, but it is interesting because, in certain systems, a good approximation is achieved by restricting the number of vacuum polarization insertions. The extreme case is quenched QED for which no such insertion is possible (with our notation, quenched QED is made of the terms $(\ell_\gamma, 0)$ for the electron propagator and $(\ell_\gamma, 1)$ for the photon propagator). The expansion over photon and electron loops can also be used in dimensional regularization, when the photon and electron is renormalized at two different scales μ and μ', as suggested by Kızılersü and collaborators [66].

For a given diagram Γ, let $\mathcal{L}_\gamma(\Gamma)$ be the number of photon loops and $\mathcal{L}_e(\Gamma)$ the number of electron loops of Γ. These numbers can be counted as follows: you remove all photon lines from Γ and you count N, the number of connected components of the remaining graph. If Γ is a photon propagator or a vacuum polarization diagram, then $\mathcal{L}_e(\Gamma) = N$, and $\mathcal{L}_\gamma(\Gamma) = V/2 - \mathcal{L}_e(\Gamma)$, where V is the number of vertices of Γ. If Γ is an electron propagator or a self-energy diagram, then the connected components are an electron line (called the baseline of Γ) and $N - 1$ electron loops, so that $\mathcal{L}_e(\Gamma) = N - 1$, and $\mathcal{L}_\gamma(\Gamma) = V/2 - \mathcal{L}_e(\Gamma)$.

For the expansion over trees we have a similar definition. If t is an electron tree, then $\mathcal{L}_\gamma(t)$ is the number of branches of t pointing to the left, and $\mathcal{L}_e(t) = |t| - \mathcal{L}_\gamma(t)$. If t is a photon tree, then $\mathcal{L}_e(t)$ is the number of branches of t pointing to the right and $\mathcal{L}_\gamma(t) = |t| - \mathcal{L}_e(t)$.

Therefore, to each Feynman diagram or tree we can associate a pair of integers (ℓ_γ, ℓ_e), where ℓ_γ is the number of photon loops and ℓ_e the number of electron loops. The corresponding map is denoted $\mathcal{L}$ (i.e. $(\ell_\gamma, \ell_e) = \mathcal{L}(\Gamma)$ or $(\ell_\gamma, \ell_e) = \mathcal{L}(t)$). Inversely, we write $L = (\ell_\gamma, \ell_e)$ to designate the sum of all self-energy or vacuum polarization diagrams or trees with ℓ_γ photon loops and ℓ_e electron loops.

For the electron propagator or self-energy we have

$(0, 0) =$ ,

$(1, 0) =$ ,

$(1, 1) =$,

$(2, 0) =$ ,

$(1, 2) =$ ,

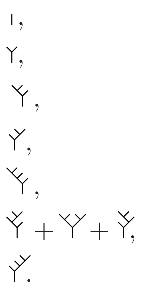
$(2, 1) =$  $+$ $+$,

$(3, 0) =$.

The reader can check this correspondence by counting the number of electron and photon loops in Sect. C.2.

For the photon propagator we have

$(0,0) =$,
$(0,1) =$,
$(0,2) =$,
$(1,1) =$,
$(0,3) =$,
$(1,2) =$ + + ,
$(2,1) =$.



The reader can check this correspondence by counting the number of electron and photon loops in Sect. C.3.

By considering the definition of the Connes–Kreimer coproduct, it can be seen that Δ_{CK} is compatible with this bigrading. For example, if $\gamma\otimes\Gamma/\gamma$ is a term of $\Delta_{CK}\Gamma$, then

$$\mathcal{L}_e(\Gamma) = \mathcal{L}_e(\gamma) + \mathcal{L}_e(\Gamma/\gamma),$$
$$\mathcal{L}_\gamma(\Gamma) = \mathcal{L}_\gamma(\gamma) + \mathcal{L}_\gamma(\Gamma/\gamma).$$

A formal way to use this bigrading is to expand all quantities over L.

$$\varphi^0_{\lambda\mu}(L;q) = \sum_{\mathcal{L}(t)=L} \varphi^0_{\lambda\mu}(t;q),$$
$$\psi^0_{\lambda\mu}(L;q) = \sum_{\mathcal{L}(t)=L} \psi^0_{\lambda\mu}(t;q),$$
$$\varphi^0(L;q) = \sum_{\mathcal{L}(t)=L} \varphi^0(t;q),$$
$$\psi^0(L;q) = \sum_{\mathcal{L}(t)=L} \psi^0(t;q).$$

Because of the definition of $\mathcal{L}(t)$, $\varphi^0(L;q)$ and $\psi^0(L;q)$ are zero if $\ell_\gamma = 0$ and $\ell_e \neq 0$. For the photon propagator, $\varphi^0_{\lambda\mu}(L;q) = 0$ is $\ell_e = 0$ and $\ell_\gamma \neq 0$. Furthermore, $\varphi^0_{\lambda\mu}(0,\ell_e;q)$ is a reducible photon diagram (a string of bubbles). Thus, $\psi^0_{\lambda\mu}(0,\ell_e;q)$ is also zero when $\ell_e > 1$. Therefore, the vacuum polarization and self-energy are

$$\Pi_{\lambda\mu}(q) = e_0^2\psi^0_{\lambda\mu}(0,1;q) + \sum_{\ell_\gamma>0,\ell_e>0} e_0^{2\ell_\gamma+2\ell_e}\psi^0_{\lambda\mu}(\ell_\gamma,\ell_e;q),$$
$$\Sigma(q) = -\sum_{\ell_\gamma>0,\ell_e\geq 0} e_0^{2\ell_\gamma+2\ell_e}\psi^0(\ell_\gamma,\ell_e;q).$$

The photon and electron Green functions are

$$D_{\lambda\mu}(q) = D^0_{\lambda\mu}(q) + \sum_{\ell_\gamma\geq 0,\ell_e>0} e_0^{2\ell_\gamma+2\ell_e}\varphi^0_{\lambda\mu}(\ell_\gamma,\ell_e;q),$$
$$S(q) = S^0(q) + \sum_{\ell_\gamma>0,\ell_e\geq 0} e_0^{2\ell_\gamma+2\ell_e}\varphi^0(\ell_\gamma,\ell_e;q),$$

with similar expressions for the renormalized quantities.

We also have an expansion of the renormalization factors:

$$Z_{2,L} = \sum_{\mathcal{L}(t)=L} \zeta_2(t),$$
$$Z_{3,L} = \sum_{\mathcal{L}(t)=L} \zeta_3(t),$$
$$Z_{m,L} = \sum_{\mathcal{L}(t)=L} \zeta_m(t).$$

The counterterms are determined by $\Sigma(q)$ and $\Pi(q)$. Thus, the non-zero terms of $Z_{i,L}$ are determined from the non-zero terms of $\psi^0(L;q)$ and $\psi^0_{\lambda\mu}(L;q)$. This gives us

$$Z_2 = 1 + \sum_{\ell_\gamma>0,\ell_e\geq 0} e^{2\ell_\gamma+2\ell_e} Z_{2,\ell_\gamma,\ell_e},$$
$$Z_3 = 1 - e^2 Z_{3,0,1} - \sum_{\ell_\gamma>0,\ell_e>0} e^{2\ell_\gamma+2\ell_e} Z_{3,\ell_\gamma,\ell_e},$$
$$\delta m = \sum_{\ell_\gamma>0,\ell_e\geq 0} e^{2\ell_\gamma+2\ell_e} Z_{m,\ell_\gamma,\ell_e}.$$

The coproduct Δ^P is now defined by

$$\Delta^P(\ell_\gamma,\ell_e) = \sum_{m_\gamma=0}^{\ell_\gamma}\sum_{m_e=0}^{\ell_e}(m_\gamma,m_e)\otimes(\ell_\gamma-m_\gamma,\ell_e-m_e).$$

This coproduct is co-commutative. The pruning operator becomes

$$P(L) = \Delta^P L - (0,0)\otimes L - L\otimes(0,0).$$

From these coproducts we derive the convolution $\star$, the reduced convolution $\bar\star$ and the antipode S_P. The unit of the algebra is (0,0).

10.1 Recursive relations

As for the case of the total number of loops, the recursive equations are obtained by summing the relations of Sect. 4.4 over the appropriate trees. For notational convenience we consider the first component, $n = 0$.

For the electron propagator (38) becomes

$$\varphi^0(\ell_\gamma+1,\ell_e;q) = \mathrm{i}\sum_{m_\gamma=0}^{\ell_\gamma}\sum_{m_e=0}^{\ell_e}\int\frac{\mathrm{d}^4p}{(2\pi)^4}S^0(q)\gamma^\lambda \times\varphi^0_{\lambda\lambda'}(m_\gamma,m_e;p)\varphi^1(\ell_\gamma-m_\gamma,\ell_e-m_e;q-p;\lambda',p).$$

In this sum, we know that $\varphi^0_{\lambda\lambda'}(m_\gamma,m_e;p) = 0$ if $m_e = 0$ and $m_\gamma \neq 0$, and $\varphi^1(\ell_\gamma-m_\gamma,\ell_e-m_e;q-p;\lambda',p) = 0$ if $m_\gamma = \ell_\gamma$ and $m_e \neq \ell_e$.

The initial data for $L = (0,0)$ are $\varphi^n(0,0;q) = \varphi^n(|;q)$ and $\varphi^n_{\lambda\mu}(0,0;q) = \varphi^n_{\lambda\mu}(|;q)$.

For the photon propagator (40) becomes

$$\varphi^0_{\mu\nu}(\ell_\gamma,\ell_e+1;q) = -\mathrm{i}\sum_{m_\gamma=0}^{\ell_\gamma}\sum_{m_e=0}^{\ell_e}\int\frac{\mathrm{d}^4p}{(2\pi)^4}D^0_{\mu\lambda}(q)$$
$$\times\mathrm{tr}\left[\gamma^\lambda\varphi^1(\ell_\gamma-m_\gamma,\ell_e-m_e;p;\lambda',-q)\right]$$
$$\times\varphi^0_{\lambda'\nu}(m_\gamma,m_e;q),$$

For the vacuum polarization, we have similarly

$$\psi^{0\mu\nu}(\ell_\gamma,\ell_e+1;q)$$
$$= -\mathrm{i}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\left[\gamma^\mu\varphi^1(\ell_\gamma,\ell_e;p;\nu,-q)\right].$$

The renormalized electron propagator is

$$\bar\varphi^0(L;q) = S_P(Z_{2,L})S^0(q) + Z_{m,L}S^0(q)^2$$
$$+S^0(q)(Z_m\bar\star\bar\varphi^0(q))(L)$$
$$+\mathrm{i}\sum_{m_\gamma=0}^{\ell_\gamma-1}\sum_{m_e=0}^{\ell_e}\int\frac{\mathrm{d}^4p}{(2\pi)^4}S^0(q)\gamma^\lambda$$
$$\times\bar\varphi^0_{\lambda\lambda'}(m_\gamma,m_e;p)$$
$$\times\bar\varphi^1(\ell_\gamma-m_\gamma-1,\ell_e-m_e;q-p;\lambda',p),$$

and the renormalized vacuum polarization

$$\bar\psi^0_{\mu\nu}(L;q) = Z_{3,L}\psi^0_{\mu\nu}(0;q) - \mathrm{i}\sum_{m_\gamma=0}^{\ell_\gamma}\sum_{m_e=0}^{\ell_e-1}\int\frac{\mathrm{d}^4p}{(2\pi)^4}$$
$$\times\mathrm{tr}\left[\gamma^\mu\bar\varphi^1(\ell_\gamma-m_\gamma,\ell_e-m_e-1;p;\nu,-q)\right]Z_{2,m_\gamma,m_e}.$$

10.2 Finiteness

If we follow the argument leading to the recursive equations for the expansion over trees, and we substitute electron and photon loops to trees, we see that these equations are uniquely determined by the Dyson relations

$$Z_3(x,y,m)\bar D(q;x,y,m) = D(q;x_0,y_0,m_0), \quad (80)$$
$$Z_2(x,y,m)\bar S(q;x,y,m) = S(q;x_0,y_0,m_0), \quad (81)$$

where

$$Z_2 = 1 + \sum_{\ell_\gamma>0,\ell_e\geq 0} x^{2\ell_\gamma}y^{2\ell_e}Z_{2,\ell_\gamma,\ell_e},$$
$$Z_3 = 1 - y^2 Z_{3,0,1} - \sum_{\ell_\gamma>0,\ell_e>0} x^{2\ell_\gamma}y^{2\ell_e}Z_{3,\ell_\gamma,\ell_e},$$
$$\delta m = \sum_{\ell_\gamma>0,\ell_e\geq 0} x^{2\ell_\gamma}y^{2\ell_e}Z_{m,\ell_\gamma,\ell_e},$$
$$m_0 = m + \delta m,$$

$$D_{\lambda\mu}(q) = D^0_{\lambda\mu}(q) + \sum_{\ell_\gamma\geq 0,\ell_e>0} x_0^{2\ell_\gamma}y_0^{2\ell_e}\varphi^0_{\lambda\mu}(\ell_\gamma,\ell_e;q),$$
$$S(q) = S^0(q) + \sum_{\ell_\gamma>0,\ell_e\geq 0} x_0^{2\ell_\gamma}y_0^{2\ell_e}\varphi^0(\ell_\gamma,\ell_e;q),$$

$$\bar D_{\lambda\mu}(q) = D^0_{\lambda\mu}(q) + \sum_{\ell_\gamma\geq 0,\ell_e>0} x^{2\ell_\gamma}y^{2\ell_e}\bar\varphi^0_{\lambda\mu}(\ell_\gamma,\ell_e;q),$$
$$\bar S(q) = S^0(q) + \sum_{\ell_\gamma>0,\ell_e\geq 0} x^{2\ell_\gamma}y^{2\ell_e}\bar\varphi^0(\ell_\gamma,\ell_e;q),$$

and

$$x_0 = x/\sqrt{Z_3(x,y,m)}, \quad (82)$$
$$y_0 = y/\sqrt{Z_3(x,y,m)}. \quad (83)$$

In practice, $x_0 = y_0 = e_0$ and $x = y = e$, but these series enable us to keep track of the number of photon and electron loops.

To show that these equations give finite results, we have to prove that they are obtained from the standard Dyson equations for the renormalization of QED Feynman diagrams. More precisely, we use the Dyson equations for the inverse of $D(q)$ and $S(q)$:

$$q_\lambda q_\mu - q^2 g_{\lambda\mu} - \xi q_\lambda q_\mu - \bar\Pi'_{\lambda\mu}(q)$$
$$= Z_3(e,m)(q_\lambda q_\mu - q^2 g_{\lambda\mu} - \xi_0 q_\lambda q_\mu - \Pi_{\lambda\mu}(q)),$$

$$\gamma\cdot q - m - \bar\Sigma(q) = Z_2(e,m)(\gamma\cdot q - m - \delta m - \Sigma(q)),$$

with

$$e_0 = \frac{Z_1 e}{Z_2\sqrt{Z_3}}. \quad (84)$$

In these equations, $\bar\Pi'_{\lambda\mu}(q)$ is given by (70), Z_1, Z_2, Z_3 and δm are given by (73), (74), (75) and (76), and finally $\bar\Sigma(q)$ is

$$\bar\Sigma(q) = \sum_{\Gamma=} e^{|\Gamma|}R(\Gamma;q). \quad (85)$$

The full proof requires the tools developed in [64], but we can indicate here its basic ingredients. The first property is that each Feynman diagram has a specific number of photon and electron loops, and that the product of two Feynman diagrams correspond to the sum of its photon and electron loops. Therefore, any term in the perturbative expansion of the Dyson equations has a well-defined number of photon and electron loops. The second property is that, for a given self-energy diagram Γ, all the reduced vertex diagrams Γ_j (deduced from Γ by inserting a photon dangling bond on each of its electron lines) have the same number of photon and electron loops as Γ. Finally, when all the terms corresponding to a given number of photon and electron loops are summed, all the Γ_j of a given Γ are present in the sum, and the term Γ will also belong to the sum to compensate for the Γ_j by (72). The last point is the main difference with the tree expansion, and it ensures that Z_1 is not required in the Dyson equations (80) and (81).

We can illustrate the last statement with an example. From the definition of Z_1 and Z_2 given in (73) and (74) we can write

$$\frac{Z_1}{Z_2} = \left(1 + \sum_{\gamma_j} e^{|\gamma_j|-1} C_1(\gamma_j)\right) \times \left(1 + \sum_{n=1}^{\infty} \sum_{\gamma^1 \ldots \gamma^n} C_2(\gamma^1)\cdots C_2(\gamma_n)\right),$$

where γ_j are reduced vertex diagrams and $\gamma^1, \ldots, \gamma^n$ are self-energy diagrams. In the right-hand side, we want to determine the term t_L with ℓ_γ photon loops and ℓ_e electron loops, so that $L = (\ell_\gamma, \ell_e)$, and we want $L \neq (0,0)$. This term is given by

$$t_L = \sum_{n=1}^{\ell_\gamma} \sum_{\mathcal{L}(\gamma^1)+\cdots\mathcal{L}(\gamma^n)=L} C_2(\gamma^1)\cdots C_2(\gamma^n) + \sum_{\mathcal{L}(\gamma_j)=L} C_1(\gamma_j) + \sum_{\mathcal{L}(\gamma_j)+\mathcal{L}(\gamma^2)+\cdots\mathcal{L}(\gamma^n)=L} C_1(\gamma_j)C_2(\gamma^2)\cdots C_2(\gamma^n).$$

In the last sum $n \geq 2$. Now the argument runs as follows: if one γ_j belongs to the last sum, then all γ_j coming from the same γ belong to it, because they have the same number of photon and electron loops; moreover, this γ is one of the γ^1 of the first sum, since γ has the same number of loops as γ_j. On the other hands, for each γ^1 in the first sum, all the γ_j corresponding to it belong to the second sum. Therefore, (72) can be used and we obtain $t_L = 0$. The same argument shows that all $C_1(\gamma_j)$ are eliminated from terms with a given number of electron and photon loops in e_0^{2n}. Hence, all $C_1(\gamma_j)$ disappear from the Dyson equation expanded over electron and photon loops. Counting the number of electron and photon loops amounts to introducing two variables x and y instead of one charge.

To summarize, we started from the Dyson equation expanded over diagrams, which contains only finite renormalized quantities, and we summed over all diagrams containing a given number of photon and electron loops. This sum gives again finite renormalized quantities and eliminates all C_1 counterterms. The absence of C_1 counterterms amounts to the use of $Z_1 = Z_2$ in the Dyson equation expanded over loops, and the expansion over loops amounts to the use of two charges x and y. Thus, the Dyson equation involving finite quantities gives exactly (81) and (80) with renormalized "charges" given by (82) and (83). Since the recursive equations determined the unique solution of these equations, we conclude that the recursive equations determine finite renormalized quantities.

11 Hopf algebra for massless QED

In this section, we determine a coproduct from the recursive equations (57), (63) and (65) and the product law (64) for massless QED. The case of massless QED is much simpler because the mass is not renormalized. Thus, for all trees t, $\zeta_m(t) = 0$.

This coproduct determines the renormalized propagators as a function of the unrenormalized ones. In this section, it will be useful to distinguish the electron and photon trees by the color of the root. A tree with a black root is written $t^\bullet$ and represents an electron propagator, a tree with a white root is written t° and represents a photon propagator. In a tree $t_l \vee t_r$, t_l is white and t_r is black. There are now two graftings operators $\vee_\bullet$ and $\vee_\circ$, so that $t_l \vee_\bullet t_r$ is a black tree and $t_l \vee_\circ t_r$ a white one.

Using a variation of Sweedler's notation, we write

$$\Delta t^\circ = \sum_{\Delta t^\circ} t^\circ_{(1)} \otimes t^\circ_{(2)}, \tag{86}$$

$$\Delta t^\bullet = \sum_{\Delta t^\bullet} t^\circ_{(1)} t^\bullet_{(1)} \otimes t^\bullet_{(2)}, \tag{87}$$

$$F(t^\bullet) = \sum_{F(t^\bullet)} t^\circ_{(1)} \otimes t^\bullet_{(2)}. \tag{88}$$

These equations mean that the coproduct of t° generates a sum of tensor products with one white tree on the left and one white tree on the right, the coproduct of $t^\bullet$ generates a sum of tensor products with one black tree and one white tree on the left and one black tree on the right; finally the coproduct[3] $F(t)$ generates a sum of tensor products with one white tree on the left and one black tree on the right. These trees can eventually be the root, which is the unit element of the algebra (the root is neither white nor black, or both, as you wish).

To avoid products of white trees in (86), (87) and (88), we took advantage of the fact that, according to (31), the $\varphi_{\mu\nu}$ of a white tree $t_l \vee_\circ t_r$ can be written as a product of $\varphi_{\mu\nu}(| \vee_\circ t_r)$ by $\varphi_{\mu\nu}(t_l)$. From (64), we also know that this property is compatible with renormalization. Therefore, we translate this property into an inner product over white trees. The product of two white trees $s^\circ t^\circ$ is defined recursively by $(s_l \vee_\circ s_r)t = (s_l t) \vee_\circ s_r$ and $|t = t$. In particular $(| \vee_\circ s)t = t \vee_\circ s$, which is what we need. Surprisingly, this product (called "over") has been discovered by Loday and Ronco in a completely different context [67]. The product is related to the product of photon diagrams by the expression

$$\varphi_{\lambda\mu}(t \vee_\circ s) = \varphi_{\lambda\mu}((| \vee_\circ s)t) = \varphi_{\lambda\nu}(| \vee_\circ s)[D^0]^{-1^{\nu\nu'}} \varphi_{\nu'\nu}(t).$$

The coproduct Δ acting on white and black trees is defined by the recursive equations

[3] More precisely, F is a coaction.

$$\Delta(|\underset{\circ}{\vee} t) = (|\underset{\circ}{\vee} t) \otimes | + \sum_{F(t)} t^\circ_{(1)} \otimes (|\underset{\circ}{\vee} t^\bullet_{(2)}), \quad (89)$$

$$\Delta(t_l \underset{\bullet}{\vee} t_r) = (t_l \underset{\bullet}{\vee} t_r) \otimes | + \sum_{\Delta t_l, \Delta t_r} (t^\circ_{l(1)} t^\circ_{r(1)}) t^\bullet_{r(1)} \otimes (t^\circ_{l(2)} \underset{\bullet}{\vee} t^\bullet_{r(2)}), \quad (90)$$

$$F(t_l \underset{\bullet}{\vee} t_r) = \sum_{\Delta t_l, F(t_r)} (t^\circ_{r(1)} t^\circ_{l(1)}) \otimes (t^\circ_{l(2)} \underset{\bullet}{\vee} t^\bullet_{r(2)}), \quad (91)$$

with the initial values $\Delta | = | \otimes |$ and $F(|) = | \otimes |$, and with the compatibility of the "over" product with renormalization: $\Delta(s^\circ t^\circ) = \Delta s^\circ \Delta t^\circ$. In particular,

$$\Delta t_l \underset{\circ}{\vee} t_r = \Delta(|\underset{\circ}{\vee} t_r) \Delta t_l.$$

With this notation, we can now write the coproduct of a general white tree

$$\Delta(t_l \underset{\circ}{\vee} t_r) = \sum_{\Delta t_l} (t^\circ_{l(1)} \underset{\circ}{\vee} t_r) \otimes t^\circ_{l(2)} + \sum_{\Delta t_l, F(t_r)} (t^\circ_{r(1)} t^\circ_{l(1)}) \otimes (t^\circ_{l(2)} \underset{\circ}{\vee} t^\bullet_{r(2)}). \quad (92)$$

These preliminaries enable us to write the relation between renormalized and unrenormalized propagators as

$$\bar\varphi^0_{\mu\nu}(t^\circ; q) = \sum_{\Delta t^\circ} \zeta(t^\circ_{(1)}) \varphi^0_{\mu\nu}(t^\circ_{(2)}; q), \quad (93)$$

$$\bar\varphi^0(t^\bullet; q) = \sum_{\Delta t^\bullet} \zeta(t^\circ_{(1)}) \zeta(t^\bullet_{(1)}) \varphi^0(t^\bullet_{(2)}; q), \quad (94)$$

$$\bar f(t; q) = \sum_{\Delta t^\bullet} \zeta(t^\circ_{(1)}) f^0(t^\bullet_{(2)}). \quad (95)$$

The general counterterm ζ is a scalar over black and white trees defined by $\zeta(|) = 1$ and

$$\zeta(t^\bullet) = \rho(t^\bullet), \quad (96)$$
$$\zeta(s^\circ t^\circ) = \zeta(s^\circ)\zeta(t^\circ),$$
$$\zeta(|\underset{\circ}{\vee} t) = \zeta_3(| \vee t).$$

In particular $\zeta(t_l \underset{\circ}{\vee} t_r) = \zeta_3(| \vee t_r)\zeta(t_l)$. We recall that $\rho(t) = \zeta_2 \circ S_P(t)$. Equations (93) and (94) are given in expanded form in Appendix 2 for trees up to order 3.

We prove this recursively. From the list of Appendix 2, (93) and (94) are satisfied for all trees up to order 3. The same can be checked for (95). Assume that they are satisfied up for trees with $2N-1$ vertices. Take a tree with $2N+1$ vertices. Take first $|\underset{\circ}{\vee} t$, then use (95) for $\bar f^1$ in (65). This yields

$$\bar\varphi^0_{\mu\nu}(| \vee t; q) = \zeta_3(| \vee t)\varphi^{\mathrm{T}}_{\mu\nu}(|; q) + \sum_{F(t)} \zeta(t_{(1)}) \varphi^0_{\mu\nu}(| \vee t_{(2)}).$$

This is (93) for the coproduct defined by (89). If we take now $t_l \underset{\circ}{\vee} t_r$, where t_l and t_r have less than $2N+1$ vertices, we can expand $\bar\varphi^0_{\mu\nu}(| \vee t_r)$ and $\bar\varphi^0_{\mu\nu}(t_l)$ over unrenormalized terms. Then, using (64) we find

$$\begin{aligned}
\bar\varphi^0_{\mu\nu}(t_l \vee t_r; q) &= \Big(\zeta_3(| \vee t_r)\varphi^{\mathrm{T}}_{\mu\nu}(|; q) \\
&\quad + \sum_{F(t_r)} \zeta(t_{r(1)}) \varphi^0_{\mu\nu}(| \vee t_{r(2)}; q)\Big) \\
&\quad \times \sum_{\Delta t_l} \zeta(t_{l(1)}) \bar\varphi^0_{\mu\nu}(t_{l(2)}; q) \\
&= \sum_{\Delta t_l} \zeta(t_{l(1)} \underset{\circ}{\vee} t_r) \bar\varphi^0_{\mu\nu}(t_{l(2)}; q) \\
&\quad + \sum_{F(t_r)\Delta t_l} \zeta(t_{r(1)} t_{l(1)}) \bar\varphi^0_{\mu\nu}(t_{l(2)} \vee t_{r(2)}; q).
\end{aligned}$$

This is (93) with the coproduct defined in (92) and the correct ζ.

For the coproduct acting on electron trees, we start from the recursive equation (57) and we use the expansion (93) for $\bar\varphi^0_{\mu\nu}(t_l; q)$ and (94) for $\bar\varphi^0(t_r; q)$. This gives us

$$\bar\varphi^0(t_r \vee t_l; q) = \rho(t_l \vee t_r)\varphi^0(|; q) + \sum_{F(t_r)\Delta t_l} \zeta(t_{l(1)} t^\circ_{r(1)}) \zeta(t^\bullet_{r(1)}) \bar\varphi^0(t_{l(2)} \vee t_{r(2)}; q).$$

The first term $\rho(t_l \vee t_r)$ is consistent with $\zeta(t_l \vee t_r)$ as defined in (96). And the other terms are consistent with (94) using the coproduct (90).

Exactly the same substitution leads to (95) using the coproduct (91).

In [64], it will be shown that Δ is co-associative and defines a Hopf algebra over the two-colored planar binary trees.

12 Conclusion

The method of Schwinger equations has a number of advantages: operator–valued distributions are avoided, as well as indefinite norms and many of the mathematically difficult concepts of quantum field theory. The methods of planar binary trees lead to recursive equations which, when summed over certain classes of trees, give recursive equations for finite renormalized quantities.

For massless QED, a Hopf algebra was explicitly defined. When all the trees of a given order ℓ are summed, we obtain a Hopf algebra over ℓ (see [64]):

$$\Delta\ell = \sum_{|t|=\ell} \Delta t.$$

This Hopf algebra over ℓ determines the renormalized (finite) quantities as a sum of divergent bare quantities multiplied by counterterms. For instance,

$$\bar\varphi^0_{\mu\nu}(\ell; q) = \sum_{\Delta\ell} \zeta(\ell_{(1)}) \varphi^0_{\mu\nu}(\ell_{(2)}; q).$$

Since the counterterms do not depend on the external field, these renormalization formulas are valid for all components:

$$\bar{\varphi}^n_{\mu\nu}(\ell; q; \{\lambda, p\}_{1,n}) = \sum_{\Delta\ell} \zeta(\ell_{(1)}) \varphi^n_{\mu\nu}(\ell^\circ_{(2)}; q; \{\lambda, p\}_{1,n}),$$

so that an infinite number of components are renormalized in one stroke.

As far as the recent and fascinating results by Connes and Kreimer [9,10] are concerned, our Hopf algebra yields a noncommutative analogue of some of their results. The study of this noncommutative Hopf algebra will be presented in detail in a forthcoming publication [64]. Finally, the noncommutative Hopf algebra of massive QED will be described in [65].

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A Appendix 1

This appendix contains some proofs.

A.1 Proof of (44) (the pruning operator is co-associative)

If $t = |$ we have $P(|) = 0$, so the identity (44) holds. So, suppose that $t \neq |$. The reason why the identity (44) holds for t is that applying the recursive definition of P, on the successive right grafters of t, at each step both sides of (44) coincide on the terms which do not involve $P(t')$ for the last right grafter t' considered. Of course, when we finally meet a right grafter t' such that $P(t') = 0$ we obtain the equality (44). We develop this idea formally.

Any tree $t \neq |$ can be written in a unique way as

$$t = t_1 \vee (t_2 \vee (... \vee (t_n \vee |)...)),$$

for some $n \leq |t| + 1$. In fact, it suffices to decompose the tree t into its left and right grafting trees, then to decompose successively the right trees as graftings of two new trees and pick up all their left sides, $t_1 := t_l$, $t_2 := (t_r)_l$, $t_3 := ((t_r)_r)_l$ and so on, until we meet an undecomposable right side $(...((t_r)_r)...)_r = |$. Since $|t| = |t_1| + |t_2| + ... + |t_n| + n - 1$ and $|t_i| \geq 0$ for all $i = 1, ..., n$, the procedure must finish for an $n \leq |t| + 1$.

Since $P(t_n \vee |) = 0$, we have

$$\begin{aligned} &P(t_{n-1} \vee (t_n \vee |)) = t_{n-1} \vee | \otimes t_n \vee |, \\ &P(t_{n-2} \vee (t_{n-1} \vee (t_n \vee |))) = t_{n-2} \vee | \otimes t_{n-1} \vee (t_n \vee |) \\ &\quad + t_{n-2} \vee (t_{n-1} \vee |) \otimes t_n \vee | \end{aligned}$$

and

$$\begin{aligned} &P(t_{n-3} \vee (t_{n-2} \vee (t_{n-1} \vee (t_n \vee |)))) \\ &\quad = t_{n-3} \vee | \otimes t_{n-2} \vee (t_{n-1} \vee (t_n \vee |)) \\ &\quad + t_{n-3} \vee (t_{n-2} \vee |) \otimes t_{n-1} \vee (t_n \vee |) \\ &\quad + t_{n-3} \vee (t_{n-2} \vee (t_{n-1} \vee |)) \otimes t_n \vee | \end{aligned}$$

Thus, for the tree $t = t_1 \vee (t_2 \vee (... \vee (t_n \vee |)...))$, we obtain

$$\begin{aligned} P(t) = &\sum_{i=1}^{n-1} t_1 \vee (t_2 \vee (... \vee (t_i \vee |)...)) \\ &\otimes t_{i+1} \vee (t_{i+2} \vee (... \vee (t_n \vee |)...)). \end{aligned}$$

Hence

$$\begin{aligned} &(P \otimes \mathrm{id}) \circ P(t) \\ &= \sum_{j=1}^{n-1} P(t_1 \vee (t_2 \vee (... \vee (t_i \vee |)...))) \\ &\otimes t_{i+1} \vee (t_{i+2} \vee (... \vee (t_n \vee |)...)) \\ &= \sum_{j=1}^{n-1} \sum_{i=1}^{j-1} t_1 \vee (...(t_i \vee |)...) \\ &\otimes t_{i+1} \vee (...(t_j \vee |)...) \otimes t_{j+1} \vee (...(t_n \vee |)...) \\ &= \sum_{1 \leq i < j \leq n-1} t_1 \vee (...(t_i \vee |)...) \\ &\otimes t_{i+1} \vee (...(t_j \vee |)...) \otimes t_{j+1} \vee (...(t_n \vee |)...), \end{aligned}$$

and similarly

$$\begin{aligned} &(\mathrm{id} \otimes P) \circ P(t) \\ &= \sum_{k=1}^{n-1} t_1 \vee (t_2 \vee (... \vee (t_k \vee |)...)) \\ &\otimes P(t_{k+1} \vee (t_{k+2} \vee (... \vee (t_n \vee |)...))) \\ &= \sum_{k=1}^{n-1} \sum_{l=k+1}^{n-1} t_1 \vee (...(t_k \vee |)...) \\ &\otimes t_{k+1} \vee (...(t_l \vee |)...) \otimes t_{l+1} \vee (...(t_n \vee |)...) \\ &= \sum_{1 \leq k < l \leq n-1} t_1(... \vee (t_k \vee |)...) \\ &\otimes t_{k+1} \vee (...(t_l \vee |)...) \otimes t_{l+1} \vee (...(t_n \vee |)...). \end{aligned}$$

Hence the identity (44) holds for any tree.

A.2 Proof of (32) and (7)

We use (31) to prove (32). We have

$$\begin{aligned}D_{\mu\nu}(q) &= \sum_t e_0^{2|t|}\varphi^0_{\mu\nu}(t;q)\\ &= \varphi^0_{\mu\nu}(\shortmid;q) + \sum_{t_1 t_2} e_0^{2|t_1|+2|t_2|+2}\varphi^0_{\mu\nu}(t_1\vee t_2;q)\\ &= \varphi^0_{\mu\nu}(\shortmid;q) + \sum_{t_1 t_2} e_0^{2|t_1|+2|t_2|+2}\varphi^0_{\mu\lambda}(\shortmid\vee t_2;q)\\ &\quad\times\psi^{0\lambda\lambda'}(\shortmid;q)\varphi^0_{\lambda'\nu}(t_1;q)\\ &= \varphi^0_{\mu\nu}(\shortmid;q) + \sum_{t_2} e_0^{2|t_2|+2}\varphi^0_{\mu\lambda}(\shortmid\vee t_2;q)\\ &\quad\times\psi^{0\lambda\lambda'}(\shortmid;q)D_{\lambda'\nu}(q).\end{aligned}$$

We multiply the last equation by $D^{-1}(q)$ on the right and by $D^{0^{-1}}(q)$ on the left. This gives us

$$[D^{0^{-1}}]_{\mu\nu}(q) = [D^{-1}]_{\mu\nu}(q) + \sum_{t_2} e_0^{2|t_2|+2}[D^{0^{-1}}]_{\mu\lambda}(q) \times \varphi^{0\lambda\lambda'}(\shortmid\vee t_2;q)\psi^0_{\lambda'\nu}(\shortmid;q).$$

Since $\varphi^{0\lambda\lambda'}(\shortmid\vee t_2;q)$ is transverse, we can replace $[D^{0^{-1}}]_{\mu\lambda}(q)$ by $\psi^0_{\mu\lambda}(\shortmid;q)$ in the above equation.

Using 30 and the definition of $\Pi_{\mu\nu}(q)$ given in (33) we obtain

$$\Pi_{\mu\nu}(q) = \sum_{t_2} e_0^{2|t_2|+2}\psi^0_{\mu\lambda}(\shortmid;q)\varphi^{0\lambda\lambda'}(\shortmid\vee t_2;q)\psi^0_{\lambda'\nu}(\shortmid;q).$$

We can rewrite this last equation as

$$\Pi_{\mu\nu}(q) = \sum_{|t|>0} e_0^{2|t|}\psi^0_{\mu\nu}(t;q) = \sum_t e_0^{2|t|+2}\psi^0_{\mu\nu}(\shortmid\vee t;q),$$

with $\psi^{0\mu\nu}(t_1\vee t_2;q)=0$ if $t_1\neq\shortmid$ and

$$\psi^0_{\mu\nu}(\shortmid\vee t;q) = \psi^0_{\mu\lambda}(\shortmid;q)\varphi^{0\lambda\lambda'}(\shortmid\vee t;q)\psi^0_{\lambda'\nu}(\shortmid;q). \tag{97}$$

Finally, we use (40) with $t_1=\shortmid$

$$\varphi^0_{\mu\nu}(\shortmid\vee t_2;q) = -\mathrm{i}D^0_{\mu\lambda}(q) \times\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\left[\gamma^\lambda\varphi^1(t_2;p;\lambda',-q)\right]D^0_{\lambda'\nu}(q).$$

We multiply this equation by $D^{0^{-1}}(q)$ on the left and on the right, and we use the fact that $\varphi^0_{\mu\nu}(\shortmid\vee t_2;q)$ is transverse to get, from (97), the relation

$$\psi^{0\mu\nu}(\shortmid\vee t;q) = -\mathrm{i}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\left[\gamma^\mu\varphi^1(t;p;\nu,-q)\right].$$

If we sum over trees t, the last equation becomes

$$\Pi^{\lambda\mu}(q) = -\mathrm{i}e_0^2\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\left[\gamma^\lambda\frac{\delta S(p)}{e_0\delta A^0_\mu(-q)}\right].$$

A.3 Proof of (53)

From the definition (47) of the self-energy ψ, we use the definition of the convolution ((42)) and of the pruning operator ((43)) to write for a tree $t=t_l\vee t_r$

$$\begin{aligned}-\psi^0(t) &= \psi^0(\shortmid)\varphi^0(t)\psi^0(\shortmid) + \psi^0(\shortmid)(\varphi^0\bar{\star}\psi^0)(t)\\ &= \psi^0(\shortmid)\varphi^0(t)\psi^0(\shortmid) + \psi^0(\shortmid)\varphi^0(t_l\vee\shortmid)\psi^0(t_r)\\ &\quad+\sum_{i=1}^{n(t_r)}\psi^0(\shortmid)\varphi^0(t_l\vee u_i)\psi^0(v_i),\end{aligned}$$

where u_i and v_i are the trees obtained by pruning t_r (i.e. $P(t_r)=\sum_i u_i\otimes v_i$).

The last equation will be transformed by using the recursive definition of φ^0 for the trees t, $t_l\vee\shortmid$ and $t_l\vee u_i$ given in (38).

Therefore, we obtain

$$\begin{aligned}\psi^0(t;q) = -\mathrm{i}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\gamma^\lambda\varphi^0_{\lambda\lambda'}(t_l;p)\\ \Big[\varphi^1(t_r;q-p;\lambda',p)\\ \times\psi^0(\shortmid;q) + \varphi^1(\shortmid;q-p;\lambda',p)\psi^0(t_r;q)\\ +\sum_{i=1}^{n(t_r)}\varphi^1(u_i;q-p;\lambda',p)\psi^0(v_i;q)\Big].\end{aligned}$$

This can also be written

$$\psi^0(t;q) = \mathrm{i}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\gamma^\lambda\varphi^0_{\lambda\lambda'}(t_l;p)g(t_r;q-p;\lambda',p),$$

where

$$\begin{aligned}g(t_r;q-p;\lambda',p) &= -\varphi^1(t_r;q-p;\lambda',p)\psi^0(\shortmid;q)\\ &\quad-(\varphi^1(q-p;\lambda',p)\bar{\star}\psi^0(q))(t_r)\\ &\quad-\varphi^1(\shortmid;q-p;\lambda',p)\psi^0(t_r;q).\end{aligned}$$

It will be useful to transform $g(t_r;q-p;\lambda',p)$. To do that, we rewrite the equation for $\psi^1(t;q;\lambda,p)$ given at the end of Sect. 6.4 in [11], so that it gives

$$\begin{aligned}&\varphi^0(\shortmid;q)\psi^1(t;q;\lambda,p)\\ &\quad= \varphi^0(\shortmid;q)\gamma^\lambda\varphi^0(t;q+p)\psi^0(\shortmid;q+p)\\ &\quad+\varphi^0(t;q)\gamma^\lambda - \varphi^1(t;q;\lambda,p)\psi^0(\shortmid;q+p)\\ &\quad+\sum_{i=1}^{n(t)}\big[\varphi^0(\shortmid;q)\gamma^\lambda\varphi^0(u_i;q+p)\psi^0(v_i;q+p)\\ &\quad-\varphi^0(u_i;q)\psi^1(v_i;q;\lambda,p)\\ &\quad-\varphi^1(u_i;q;\lambda,p)\psi^0(v_i;q+p)\big].\end{aligned}$$

Now, we replace $\varphi^0(t;q+p)$ by its value given from (48), we add $g(t;q;\lambda,p)$ on both sides and we reorder a bit:

$$\begin{aligned}g(t;q;\lambda,p) &= \varphi^0(\shortmid;q)\psi^1(t;q;\lambda,p) - \varphi^0(t;q)\gamma^\lambda\\ &\quad+\varphi^0(\shortmid;q)\gamma^\lambda\varphi^0(t;q+p)\psi^0(\shortmid;q+p)\\ &\quad-\varphi^1(\shortmid;q;\lambda,p)\psi^0(\shortmid;q+p)\\ &\quad+\sum_{i=1}^{n(t)}\varphi^0(u_i;q+p)\psi^1(v_i;q;\lambda,p).\end{aligned}$$

Finally, we note that $\varphi^1(|;q;\lambda,p)=\varphi^0(|;q)\gamma^\lambda\varphi^0(t;q+p)$ and we obtain

$$\begin{aligned} g(t;q;\lambda,p) &= \varphi^0(|;q)\psi^1(t;q;\lambda,p) \\ &+ \varphi^0(t;q)\psi^1(|;q;\lambda,p) \\ &+ \sum_{i=1}^{n(t)} \varphi^0(u_i;q)\psi^1(v_i;q;\lambda,p). \end{aligned}$$

B Appendix 2

In this appendix, we collect the relation between bare and renormalized photon and electron φ up to three loops.

B.1 Electron Green function for massless QED

B.1.1 One loop

$$\bar\varphi(\mathrm{Y}) = \varphi(\mathrm{Y}) - \zeta_2(\mathrm{Y})\varphi(|).$$

B.1.2 Two loops

$$\begin{aligned} \bar\varphi(\mathrm{Y}\vee|) &= \varphi(\mathrm{Y}\vee|) \\ &+ \zeta_3(\mathrm{Y})\varphi(\mathrm{Y}) - \zeta_2(\mathrm{Y}\vee|)\varphi(|), \\ \bar\varphi(|\vee\mathrm{Y}) &= \varphi(|\vee\mathrm{Y}) \\ &- \zeta_2(\mathrm{Y})\varphi(\mathrm{Y}) - \zeta_2(|\vee\mathrm{Y})\varphi(|) + \zeta_2(\mathrm{Y})^2\varphi(|). \end{aligned}$$

B.1.3 Three loops

$$\begin{aligned} \bar\varphi((\mathrm{Y}\vee|)\vee|) &= \varphi((\mathrm{Y}\vee|)\vee|) + 2\zeta_3(\mathrm{Y})\varphi(\mathrm{Y}\vee|) \\ &+ \zeta_3(\mathrm{Y})^2\varphi(\mathrm{Y}) - \zeta_2((\mathrm{Y}\vee|)\vee|)\varphi(|), \\ \bar\varphi((|\vee\mathrm{Y})\vee|) &= \varphi((|\vee\mathrm{Y})\vee|) + \zeta_3(|\vee\mathrm{Y})\varphi(\mathrm{Y}) \\ &- \zeta_2((|\vee\mathrm{Y})\vee|)\varphi(|), \\ \bar\varphi(\mathrm{Y}\vee\mathrm{Y}) &= \varphi(\mathrm{Y}\vee\mathrm{Y}) + \zeta_3(\mathrm{Y})\varphi(|\vee\mathrm{Y}) \\ &- \zeta_2(\mathrm{Y})\varphi(\mathrm{Y}\vee|) - \zeta_2(\mathrm{Y})\zeta_3(\mathrm{Y})\varphi(\mathrm{Y}) \\ &- \zeta_2(\mathrm{Y}\vee\mathrm{Y})\varphi(|) + \zeta_2(\mathrm{Y}\vee|)\zeta_2(\mathrm{Y})\varphi(|), \\ \bar\varphi(|\vee(\mathrm{Y}\vee|)) &= \varphi(|\vee(\mathrm{Y}\vee|)) + \zeta_3(\mathrm{Y})\varphi(|\vee\mathrm{Y}) - \zeta_2(\mathrm{Y}\vee|)\varphi(\mathrm{Y}) \\ &-\zeta_2(|\vee(\mathrm{Y}\vee|))\varphi(|) + \zeta_2(\mathrm{Y}\vee|)\zeta_2(\mathrm{Y})\varphi(|), \\ \bar\varphi(|\vee(|\vee\mathrm{Y})) &= \varphi(|\vee(|\vee\mathrm{Y})) - \zeta_2(\mathrm{Y})\varphi(|\vee\mathrm{Y}) - \zeta_2(|\vee\mathrm{Y})\varphi(\mathrm{Y}) \\ &+ \zeta_2(\mathrm{Y})^2\varphi(\mathrm{Y}) + [-\zeta_2(|\vee(|\vee\mathrm{Y})) \\ &+ 2\zeta_2(|\vee\mathrm{Y})\zeta_2(\mathrm{Y}) - \zeta_2(\mathrm{Y})^3]\varphi(|). \end{aligned}$$

B.2 Photon Green function for massless QED

B.2.1 One loop

$$\bar\varphi_{\lambda\mu}(\mathrm{Y}) = \varphi_{\lambda\mu}(\mathrm{Y}) + \zeta_3(\mathrm{Y})\varphi_{\lambda\mu}(|).$$

B.2.2 Two loops

$$\begin{aligned} \bar\varphi_{\lambda\mu}(\mathrm{Y}\vee|) &= \varphi_{\lambda\mu}(\mathrm{Y}\vee|) + 2\zeta_3(\mathrm{Y})\varphi_{\lambda\mu}(\mathrm{Y}) \\ &+ \zeta_3(\mathrm{Y})^2\varphi_{\lambda\mu}(|), \\ \bar\varphi_{\lambda\mu}(|\vee\mathrm{Y}) &= \varphi_{\lambda\mu}(|\vee\mathrm{Y}) + \zeta_3(|\vee\mathrm{Y})\varphi_{\lambda\mu}(|). \end{aligned}$$

B.2.3 Three loops

$$\begin{aligned} \bar\varphi_{\lambda\mu}((\mathrm{Y}\vee|)\vee|) &= \varphi_{\lambda\mu}((\mathrm{Y}\vee|)\vee|) + 3\zeta_3(\mathrm{Y})\varphi_{\lambda\mu}(\mathrm{Y}\vee|) \\ &+ 3\zeta_3(\mathrm{Y})^2\varphi_{\lambda\mu}(\mathrm{Y}) + \zeta_3(\mathrm{Y})^3\varphi_{\lambda\mu}(|), \\ \bar\varphi_{\lambda\mu}((|\vee\mathrm{Y})\vee|) &= \varphi_{\lambda\mu}((|\vee\mathrm{Y})\vee|) + \zeta_3(\mathrm{Y})\varphi_{\lambda\mu}(|\vee\mathrm{Y}) \\ &+ \zeta_3(|\vee\mathrm{Y})\varphi_{\lambda\mu}(\mathrm{Y}) \\ &+ \zeta_3(|\vee\mathrm{Y})\zeta_3(\mathrm{Y})\varphi_{\lambda\mu}(|), \\ \bar\varphi_{\lambda\mu}(\mathrm{Y}\vee\mathrm{Y}) &= \varphi_{\lambda\mu}(\mathrm{Y}\vee\mathrm{Y}) \\ &+ \zeta_3(\mathrm{Y})\varphi_{\lambda\mu}(|\vee\mathrm{Y}) \\ &+ \zeta_3(|\vee\mathrm{Y})\varphi_{\lambda\mu}(\mathrm{Y}) \\ &+ \zeta_3(|\vee\mathrm{Y})\zeta_3(\mathrm{Y})\varphi_{\lambda\mu}(|), \\ \bar\varphi_{\lambda\mu}(|\vee(\mathrm{Y}\vee|)) &= \varphi_{\lambda\mu}(|\vee(\mathrm{Y}\vee|)) + \zeta_3(\mathrm{Y})\varphi_{\lambda\mu}(|\vee\mathrm{Y}) \\ &+ \zeta_3(|\vee(\mathrm{Y}\vee|))\varphi_{\lambda\mu}(|), \\ \bar\varphi_{\lambda\mu}(|\vee(|\vee\mathrm{Y})) &= \varphi_{\lambda\mu}(|\vee(|\vee\mathrm{Y})) + \zeta_3(|\vee(|\vee\mathrm{Y}))\varphi_{\lambda\mu}(|). \end{aligned}$$

B.3 Electron self-energy for massless QED

B.3.1 One loop

$$\bar\psi(\mathrm{Y}) = \psi(\mathrm{Y}) + \zeta_2(\mathrm{Y})\psi(|).$$

B.3.2 Two loops

$$\begin{aligned} \bar\psi(\mathrm{Y}\vee|) &= \psi(\mathrm{Y}\vee|) + \zeta_3(\mathrm{Y})\psi(\mathrm{Y}) \\ &+ \zeta_2(\mathrm{Y}\vee|)\psi(|), \\ \bar\psi(|\vee\mathrm{Y}) &= \psi(|\vee\mathrm{Y}) + \zeta_2(\mathrm{Y})\psi(\mathrm{Y}) \\ &+ \zeta_2(|\vee\mathrm{Y})\psi(|). \end{aligned}$$

B.3.3 Three loops

$$\begin{aligned} \bar\psi((\mathrm{Y}\vee|)\vee|) &= \psi((\mathrm{Y}\vee|)\vee|) + 2\zeta_3(\mathrm{Y})\psi(\mathrm{Y}\vee|) \\ &+ \zeta_3(\mathrm{Y})^2\psi(\mathrm{Y}) + \zeta_2((\mathrm{Y}\vee|)\vee|)\psi(|), \\ \bar\psi((|\vee\mathrm{Y})\vee|) &= \psi((|\vee\mathrm{Y})\vee|) + \zeta_3(|\vee\mathrm{Y})\psi(\mathrm{Y}) \\ &+ \zeta_2((|\vee\mathrm{Y})\vee|)\psi(|), \end{aligned}$$

$$
\begin{aligned}
\bar\psi(\Upsilon\vee\Upsilon) &= \psi(\Upsilon\vee\Upsilon) + \zeta_3(\Upsilon)\psi(|\vee\Upsilon) \\
&\quad + \zeta_2(\Upsilon\vee|)\psi(\Upsilon) + \zeta_2(\Upsilon\vee\Upsilon)\psi(|),\\
\bar\psi(|\vee(\Upsilon\vee|)) &= \psi(|\vee(\Upsilon\vee|)) \\
&\quad + \zeta_3(\Upsilon)\psi(|\vee\Upsilon) + \zeta_2(|\vee(\Upsilon\vee|))\psi(|) \\
&\quad + \zeta_2(\Upsilon)\psi(\Upsilon\vee|) + \zeta_2(\Upsilon)\zeta_3(\Upsilon)\psi(\Upsilon),\\
\bar\psi(|\vee(|\vee\Upsilon)) &= \psi(|\vee(|\vee\Upsilon)) + \zeta_2(\Upsilon)\psi(|\vee\Upsilon) \\
&\quad + \zeta_2(|\vee\Upsilon)\psi(\Upsilon) + \zeta_2(|\vee(|\vee\Upsilon))\psi(|).
\end{aligned}
$$

B.4 Vacuum polarization for massless QED

B.4.1 One loop

$$
\bar\psi_{\lambda\mu}(\Upsilon) = \psi_{\lambda\mu}(\Upsilon) + \zeta_3(\Upsilon)\psi_{\lambda\mu}(|).
$$

B.4.2 Two loops

$$
\begin{aligned}
\bar\psi_{\lambda\mu}(\Upsilon\vee|) &= 0,\\
\bar\psi_{\lambda\mu}(|\vee\Upsilon) &= \psi_{\lambda\mu}(|\vee\Upsilon) + \zeta_3(|\vee\Upsilon)\psi_{\lambda\mu}(|).
\end{aligned}
$$

B.4.3 Three loops

$$
\begin{aligned}
\bar\psi_{\lambda\mu}((\Upsilon\vee|)\vee|) &= 0,\\
\bar\psi_{\lambda\mu}((|\vee\Upsilon)\vee|) &= 0,\\
\bar\psi_{\lambda\mu}(\Upsilon\vee\Upsilon) &= 0,\\
\bar\psi_{\lambda\mu}(|\vee(\Upsilon\vee|)) &= \psi_{\lambda\mu}(|\vee(\Upsilon\vee|)) + \zeta_3(\Upsilon)\psi_{\lambda\mu}(|\vee\Upsilon)\\
&\quad + \zeta_3(|\vee(\Upsilon\vee|))\psi_{\lambda\mu}(|),\\
\bar\psi_{\lambda\mu}(|\vee(|\vee\Upsilon)) &= \psi_{\lambda\mu}(|\vee(|\vee\Upsilon)) + \zeta_3(|\vee(|\vee\Upsilon))\psi_{\lambda\mu}(|).
\end{aligned}
$$

C Appendix 3: The first trees

This appendix gives the Feynman diagrams corresponding to the first trees for the electron and photon Green functions.

C.1 Recursive generation of Feynman diagrams

Let $\pi^e : Y \to \mathcal{F}^e$ be the morphism which associates a sum of Feynman diagrams to each electron tree t, and $\pi^\gamma : Y \to \mathcal{F}^\gamma$ the morphism which associates a sum of Feynman diagrams to each photon tree t. In the following sections, π^e and π^γ will be given for a few trees. An alternative method to generate all the Feynman diagrams of the electron and photon propagators was given by Bachmann and collaborators [68].

We first give a recursive definition of $\pi^e \circ \varphi^0(t)$. As an example, we start with $t_r = \Upsilon$ so that $\pi^e \circ \varphi^0(t_r) =$

.

The first step is to build $\pi^e \circ \varphi^1(t_r)$ which is obtained by branching a photon dangling bond to each of the free electron lines of $\pi^e \circ \varphi^0(t_r)$ and by summing the results. For our example

$\pi^e \circ \varphi^1(\Upsilon) =$

Then, $\pi^e \circ \varphi^0(t_l \vee t_r)$ is obtained by branching the photon propagator diagrams of $\pi^\gamma \circ \varphi^0_{\lambda\mu}(t_l)$ to the dangling bonds of $\pi^e \circ \varphi^1(t_r)$ and connecting these photon propagators to the first dot of the Feynman diagrams. Then the results are multiplied on the left by a free electron line.

For our example $t_r = \Upsilon$, the reader could take $t_l = |$ to build $\pi^e \circ \varphi^0(|\vee\Upsilon)$ and take $t_l = \Upsilon$ to build $\pi^e \circ \varphi^0(\Upsilon\vee\Upsilon)$, and check the results given in the following tables.

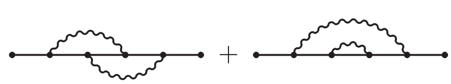
The recursive definition of $\pi^\gamma \circ \varphi^0_{\lambda\mu}(t_l \vee t_r)$ is still simpler. We also start from the diagrams generated by $\pi^e \circ \varphi^1(t_r)$; we close the two free electron lines on a new vertex (on the left of the diagram), to which we branch a free photon propagator, and the photon propagator $\pi^\gamma \circ \varphi^0_{\lambda\mu}(t_l)$ is connected to the photon dangling bond on the right of the diagrams. Again, the reader can take $t_l = |$ and $t_l = \Upsilon$ to check the sum of Feynman diagrams given for $\pi^\gamma \circ \varphi^0_{\lambda\mu}(|\vee\Upsilon)$ and $\pi^\gamma \circ \varphi^0_{\lambda\mu}(\Upsilon\vee\Upsilon)$ in the following tables.

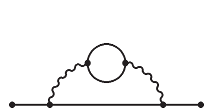
C.2 Electron Green function

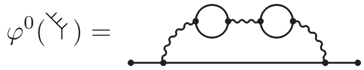
For the electron Green functions, all electron loops are oriented anticlockwise and the propagator is oriented from right to left, as indicated in $\varphi^0(|)$. Notice that the last two diagrams of $\varphi^0(|\vee(\Upsilon\vee|))$ are zero by Furry's theorem. However, they are useful to generate Feynman diagrams for higher order trees. The Feynman diagrams of $\psi^0(t)$ for $t \neq |$ are obtained from those of $\varphi^0(t)$ by keeping only the 1PI diagrams of $\varphi^0(t)$, removing the first and last free electron lines, and by putting a minus sign in front of all diagrams.

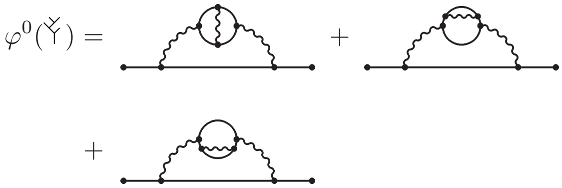
$\varphi^0(|) =$

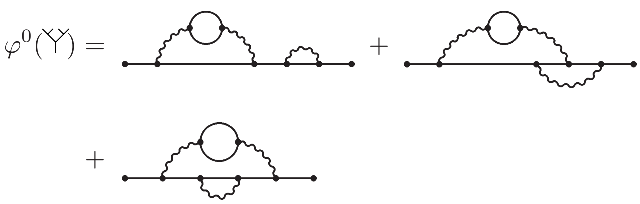
$\varphi^0(\Upsilon) =$ 

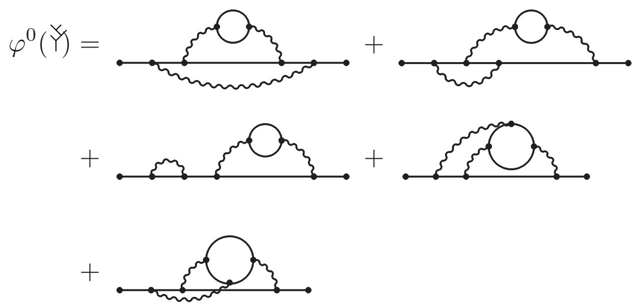
$\varphi^0(|\vee\Upsilon) =$ 

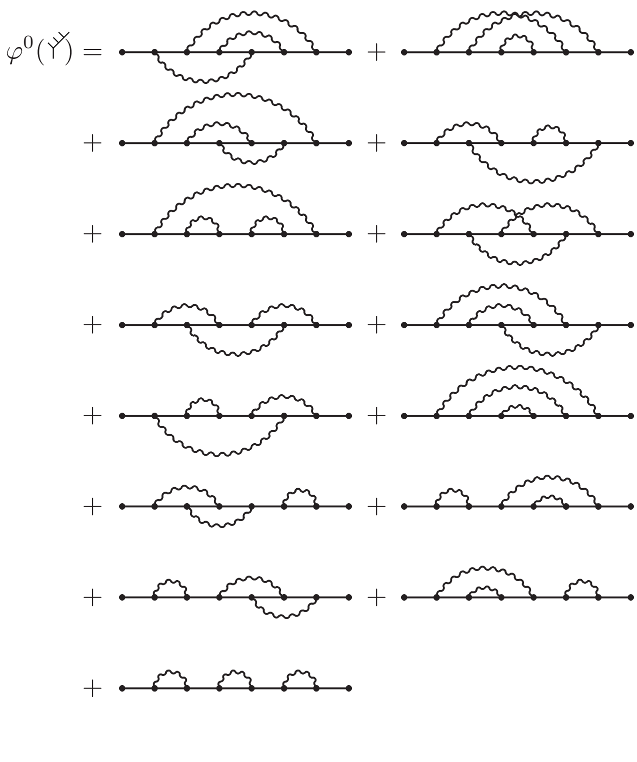
$\varphi^0(\Upsilon\vee|) =$ 





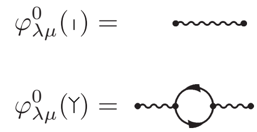


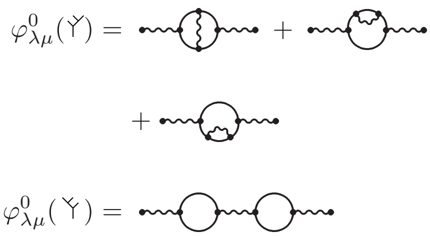


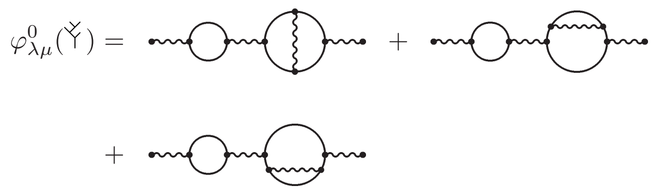


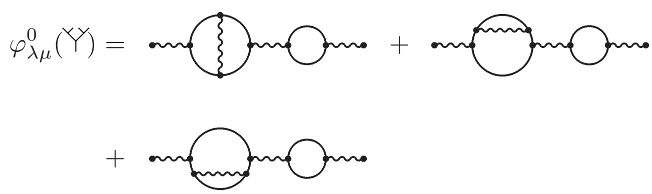
C.3 Photon Green function

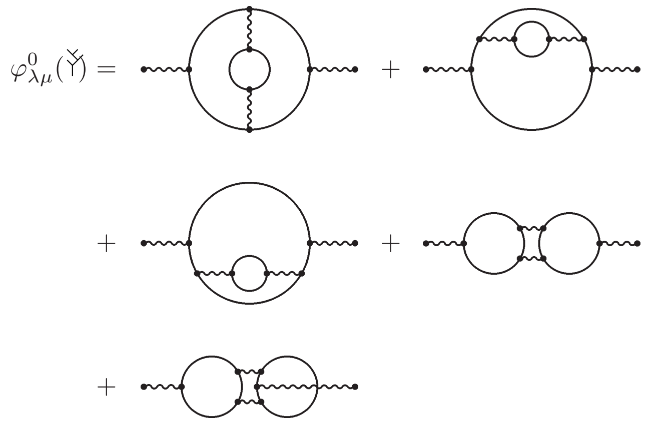
For the photon Green functions, all electron loops are oriented anticlockwise. This is only indicated explicitly for $\varphi^0_{\lambda\mu}(\text{Y})$. The last two diagrams of $\varphi^0_{\lambda\mu}(\text{Y})$ are zero by Furry's theorem. However, they are useful to generate Feynman diagrams for higher order trees. The Feynman diagrams of $\psi^0_{\lambda\mu}(t)$ for $t \neq |$ are obtained from those of $\varphi^0_{\lambda\mu}(t)$ by keeping only the 1PI diagrams of $\varphi^0_{\lambda\mu}(t)$ and removing the first and last free photon lines.

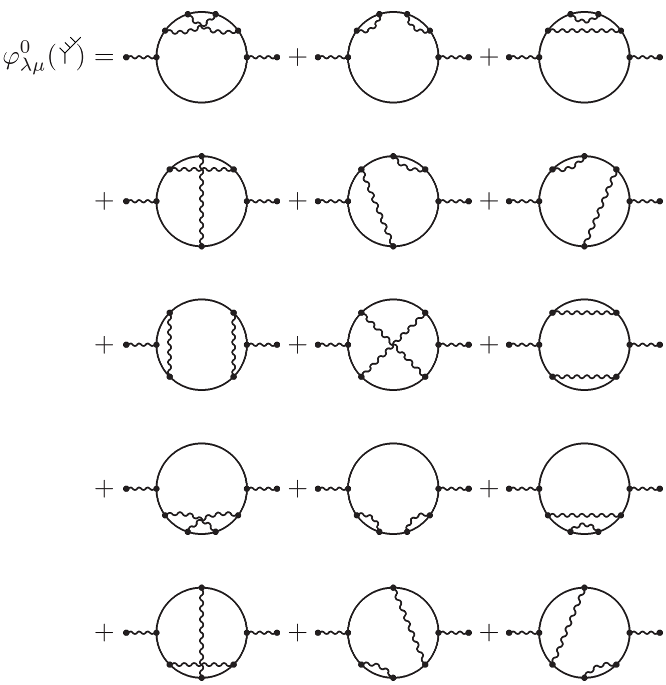












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